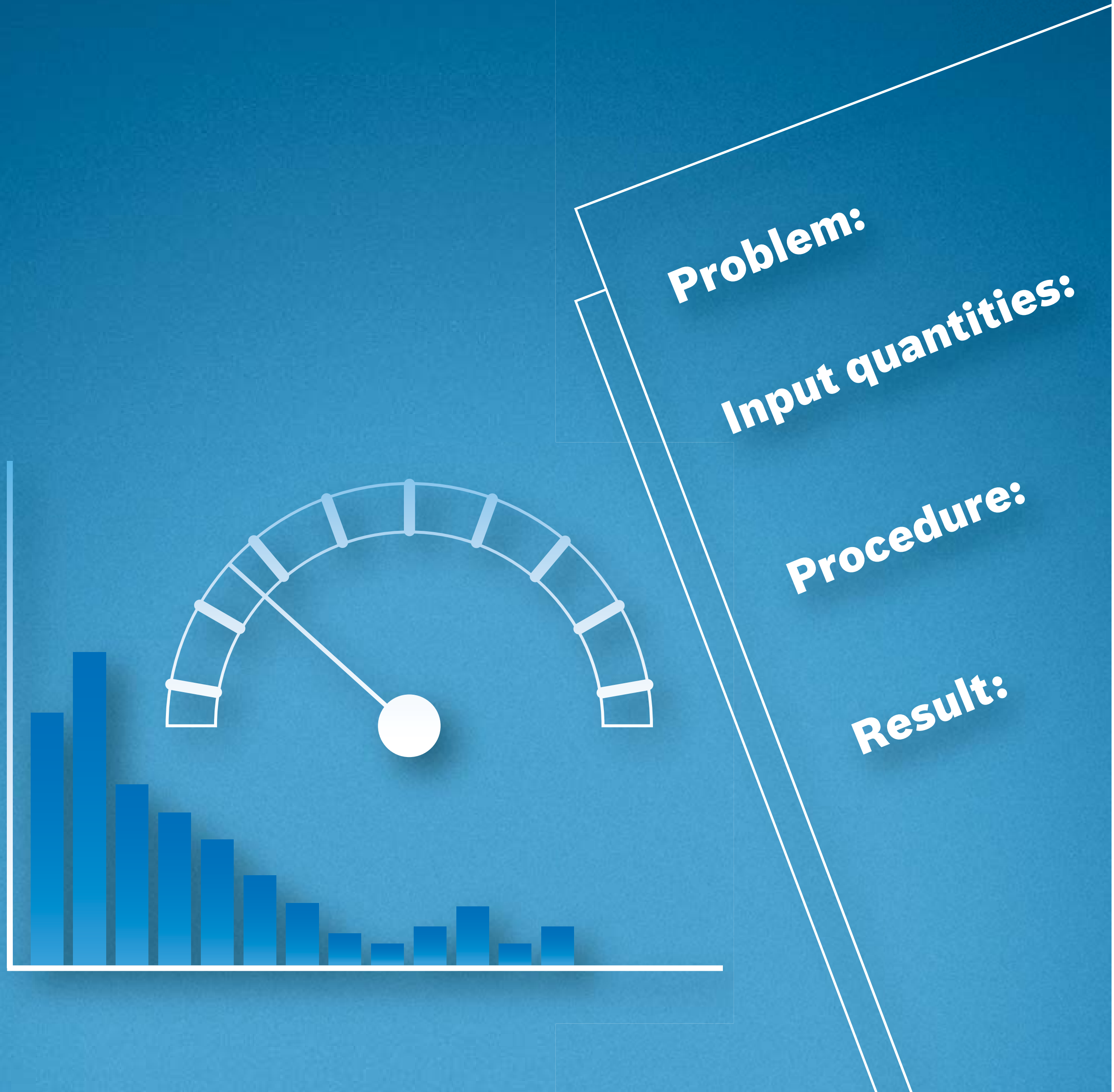

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Quality Management in the Bosch Group | Technical Statistics

4. Statistical Procedures
Formulas and Tables

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Quality Management in the Bosch Group
Technical Statistics

Booklet 4
Statistical Procedures — Formulas and Tables

Edition 01.2016

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Contents:

Tables:

Foreword

The booklet at hand contains selected procedures, statistical tests and practical examples which are found in this or a similar form time and again in the individual phases of product development. They represent a cross-section of applied technical statistics through the diverse fields, starting with a calculation of statistical characteristics and continuing on to secondary methods within the scope of statistical experimental planning and assessment.

The majority of procedures is represented one to each page and explained on the basis of examples from Bosch practice. Usually only a few input quantities which are immediately present are necessary for their application, e.g. the measurement values themselves, their number, maximum and minimum, or statistical characteristics which can be easily calculated from the measurement data.

For reasons of clarity, a more in-depth treatment of the respective theoretical correlations is consciously avoided. It should be observed, however, that the normal distribution as the distribution of the considered population is presupposed in the procedures represented in the following: Outlier test, confidence interval for μ and σ, F Test, t Test, calculation of the minimum sample size, simple and factorial variance analyses.

In several examples, the calculated intermediate result is rounded. Through further calculation with the rounded value, slightly different numbers can be produced in the end result depending on the degree of rounding. These small differences are insignificant, however, to the statistical information hereby gained.

Fundamentally, specific statistical statements are always coupled with a selected confidence level or level of significance. It should be clear in this association that the allocation of the attributes “significant” and “insignificant” to the probabilities $\geq 95\%$ or $\geq 99\%$ were arbitrarily chosen in this booklet. While a statistical level of significance of 1 % can seem completely acceptable to a SPC user, the same level of significance is presumably still much too large for a development engineer who is responsible for a safety-related component.

In connection with statistical tests, a certain basic example is always recognizable in the process. The test should enable a decision between a so-called null hypothesis and an alternative hypothesis. Depending on the choice, one is dealing with a one-sided or two-sided question. This must, in general, be considered in the determination of the tabulated test statistic belonging to the test. In this booklet, the tables are already adjusted to the question with the respective corresponding process so that no problems occur in these regards for the user.

Finally, the banal-sounding fact that each statistical process “cannot take more information from a data set” than that which is contained in it (usually, the sample size, among other factors, limits the information content) must be emphasized. Moreover, a direct data analysis of measurement values prepared in a graphically sensible manner is usually easier to implement and more comprehensible than a statistical test.

Statistical procedures are auxiliary media which can support the user in the data preparation and assessment. They cannot, however, take the place of sound judgement in decision making.

Arithmetic Mean

The arithmetic mean $\bar{x}$ is a characteristic for the position of a set of values x_i on the number line (x-axis).

Input quantities:

x_i, n

Formula:

$$\bar{x} = \frac{1}{n} \cdot (x_1 + x_2 + \ldots + x_n)$$

Notation with sum sign:

$$\bar{x} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$$

REMARK:

The arithmetic mean of a sample is frequently viewed as an estimated value for the unknown mean μ of the population of all values upon which the sample is based. It is to be observed that the arithmetic mean of a data set can be greatly modified by a single outlier. In addition, in a skewed distribution for example, many more individual values may lie below the mean than above it.

EXAMPLES:

The means of the following measurement series should be determined:

Measurement series 1	*47*	*45*	*57*	*44*	*47*	*46*	*58*	*45*	*46*	*45*
Measurement series 2	*48*	*49*	*45*	*50*	*49*	*47*	*47*	*48*	*49*	*48*
Measurement series 3	*53*	*46*	*51*	*44*	*50*	*45*	*45*	*51*	*50*	*45*

Measurement series 1: $\bar{x} = \frac{47+45+57+44+47+46+58+45+46+45}{10} = \frac{480}{10} = 48$

Measurement series 2: $\bar{x} = \frac{48+49+45+50+49+47+47+48+49+48}{10} = \frac{480}{10} = 48$

Measurement series 3: $\bar{x} = \frac{53+46+51+44+50+45+45+51+50+45}{10} = \frac{480}{10} = 48$

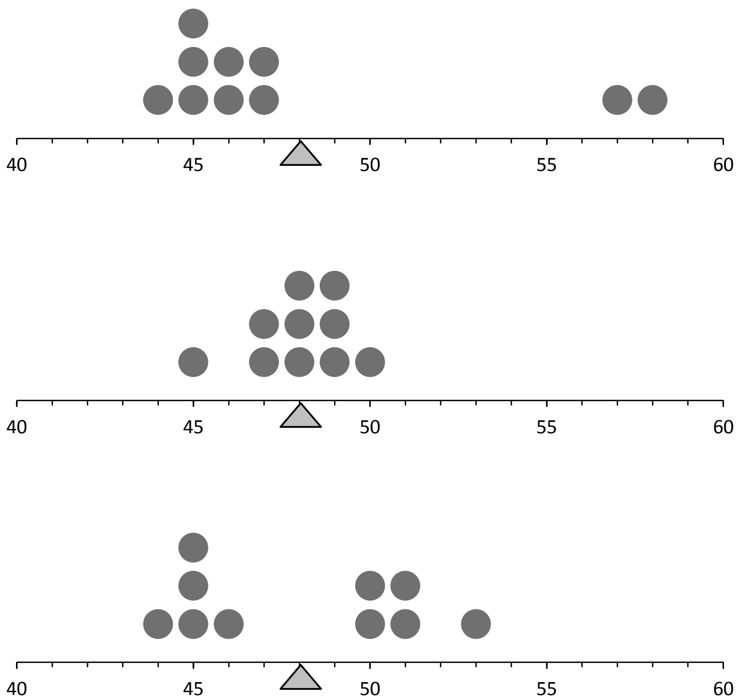

Fig. 1: Dot diagrams of the three measurement series

The schematic representation of the balance bar should illustrate that the arithmetic mean corresponds to the respective center of gravity of the weights which lie at position x_i*. If the balance bar is supported at position* $\bar{x}$*, then the balance finds itself in equilibrium.*

On the basis of the first measurement series, it becomes clear that the arithmetic mean of extreme values (e.g. also outliers) can be greatly influenced. In this example, only two values lie above the mean.

As the third measurement series shows, it is possible that no values at all will lie in the proximity of the mean.

Median

The median $\tilde{x}$ is, like the arithmetic mean $\bar{x}$, a characteristic for the position of a set of values x_i on the number line (x-axis).

Input quantities:

x_i, n

Procedure:

1. The values $x_1, x_2, \ldots, x_n$ are arranged in order of magnitude:

   $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$

   $x_{(1)}$ is the smallest, and $x_{(n)}$ is the largest value.

2. Determination of the median:

   $\tilde{x} = x_{\left(\frac{n+1}{2}\right)}$ if n is odd

   $\tilde{x} = \dfrac{x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}}{2}$ if n is even

The median, then, is the value which lies in the middle of the sequentially arranged group of numbers. If n is an even number (divisible by 2 without remainder), there will be no number in the "middle". The median is then equal to the mean of the two middle numbers.

REMARK:

On the basis of the definition, the same number of values of the data set always lies below and above the median $\tilde{x}$. The median is therefore preferred to the arithmetic mean with unimodal, skewed distributions.

EXAMPLES:

The operation "shaft grinding" is monitored within the scope of statistical process control (SPC) with the assistance of an $\tilde{x}$-R chart. In the last 5-measurement sample, the following deviations of the monitored characteristic from the nominal dimension were measured :

4.8 5.1 4.9 5.2 4.5.

The median of the sample is to be recorded on the quality control chart.

Well-ordered values: 4.5 4.8 4.9 5.1 5.2

$\tilde{x} = 4{,}9$

In the test of overpressure valves, the following opening pressures (in bar) were measured:

10.2 10.5 9.9 14.8 10.6 10.2 13.9 9.7 10.0 10.4.

Arrangement of the measurement values:

9.7 9.9 10.0 10.2 10.2 10.4 10.5 10.6 13.9 14.8

$$\tilde{x} = \frac{10.2 + 10.4}{2} = 10.3$$

Standard Deviation

The standard deviation s is a characteristic for the dispersion in a group of values x_i, with respect to the arithmetic mean $\bar{x}$. The square s^2 of the standard deviation s is called the variance.

Input quantities:

x_i, $\bar{x}$, n

Formula:

$$s = \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Alternatively, the following formulas can also be used for the calculation of s:

$$s = \sqrt{\frac{1}{n-1} \cdot \left[\sum_{i=1}^{n} x_i^2 - \frac{1}{n} \cdot \left(\sum_{i=1}^{n} x_i \right)^2 \right]}$$

$$s = \sqrt{\frac{1}{n-1} \cdot \left[\sum_{i=1}^{n} x_i^2 - n \cdot \bar{x}^2 \right]}$$

REMARK:

The manner of calculation of s *is independent of the distribution from which the values* x_i *originate and is always the same. Even a pocket calculator with statistical functions does not "know" anything about the distribution of the given values. It always calculates corresponding to one of the formulas given above.*

The representation on the right side shows the probability density function of the standard normal distribution, a triangular distribution and a rectangular distribution, which all possess the same theoretical standard deviation $\sigma = 1$*. If one takes adequately large samples from these distributions and calculates the respective standard deviation* s*, these three values will only differ slightly from each other, even though the samples can clearly have different ranges.*

The standard deviation s *should, then, always be considered in connection with the distribution upon which it is based.*

EXAMPLE:

The standard deviation s *of the following data set should be calculated:*

6.1 5.9 5.4 5.5 4.8 5.9 5.7 5.3.

Many pocket calculators offer the option of entering the individual values with the assistance of a data key and calculating the standard deviation by pressing the s_x *key and displaying it (these keys can be different depending on the calculator).*

Calculation with the second given formula:

x_i	x_i^2
6.1	37.21
5.9	34.81
5.4	29.16
5.5	30.25
4.8	23.04
5.9	34.81
5.7	32.49
5.3	28.09

$$\sum_{i=1}^{8} x_i = 44.6 \qquad \left(\sum_{i=1}^{8} x_i\right)^2 = (44.6)^2 = 1989.16 \qquad \sum_{i=1}^{8} x_i^2 = 249.86$$

Insertion into the second formula produces: $s = \sqrt{\frac{1}{7} \cdot \left(249.86 - \frac{1989.16}{8}\right)} = 0.4166$.

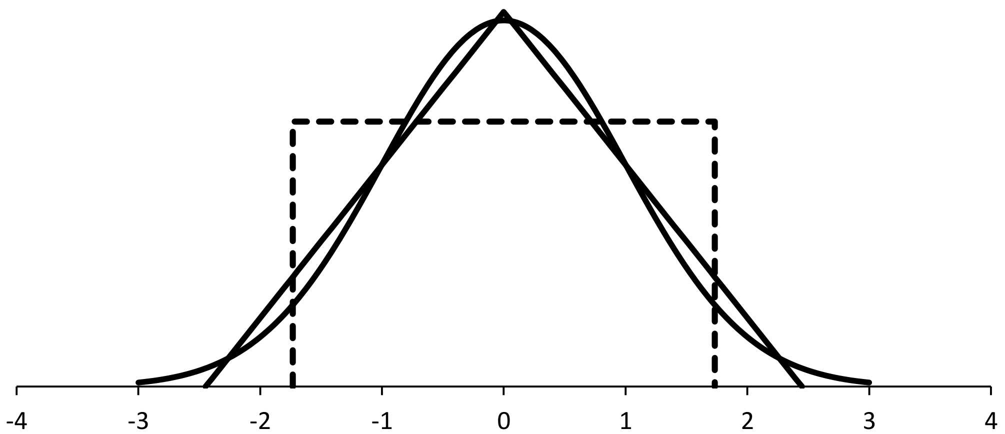

Fig. 2: Probability density functions of various distributions with the same standard deviation $\sigma = 1$ (see remark on the left side)

Histogram

If the number line (x-axis) is divided into individual, adjacent fields in which one gives the boundaries, then this is referred to as a classification. Through the sorting of the values of a data set into the individual classes (grouping), a number of values which fall into a certain class is produced for each class. This number is called the absolute frequency.

If the absolute frequency n_j is divided respectively by the total number n of all values, the relative frequencies h_j are obtained. A plot of these relative frequencies over the classes in the form of adjacently lined-up rectangles is called a histogram. The histogram conveys an image of the value distribution.

For the "appearance" of the histogram, the choice of the classification can be of decisive significance. There is no uniform, rigid rule, however, for the determination of the classification, but rather merely recommendations which are listed in the following. Lastly, one must orient oneself to the individual peculiarities of the current problem during the creation of a histogram.

Problem:

On the basis of the n individual values x_i of a data set, a histogram is to be created.

Input quantities:

x_i, x_{max}, x_{min}, n

Procedure:

1. Select a suitable classification

   Determine the number k of classes

   Rule of thumb: $25 \leq n \leq 100$ $\quad k = \sqrt{n}$

   $n > 100$ $\quad k = 5 \cdot \log(n)$

   The classification should be selected with a fixed class width (if possible) so that "simple numbers" are produced as class limits. The first class should not be open to the left and the last should not be open to the right. Empty classes, i.e. those into which no values of the data set fall, are to be avoided.
2. Sort the values x_i into the individual classes

   Determine the absolute frequency n_j for $j = 1, 2, \ldots, k$
3. Calculate the relative frequency $h_j = \frac{n_j}{n}$ for $j = 1, 2, \ldots, k$
4. Plot the relative frequencies h_j (y-axis) over the individual classes of the classification (x-axis)

REMARK:

The recommendation of calculating the class width b according to $b = \frac{x_{max} - x_{min}}{k-1}$ usually results in class limits with several decimal places, which is impractical for the manual creation of a histogram and, beyond that, can lead to empty classes.

EXAMPLE:

Relays were tested for series production start-up. A functionally decisive characteristic is the so-called "response voltage". With 50 relays, the following values of the response voltage U_{resp} were measured in volts. A histogram should be created.

6.2	6.5	6.1	6.3	5.9	6.0	6.0	6.3	6.2	6.4
6.5	5.5	5.7	6.2	5.9	6.5	6.1	6.6	6.1	6.8
6.2	6.4	5.8	5.6	6.2	6.1	5.8	5.9	6.0	6.1
6.0	5.7	6.5	6.2	5.6	6.4	6.1	6.3	6.1	6.6
6.4	6.3	6.7	5.9	6.6	6.3	6.0	6.0	5.8	6.2

Corresponding to the rule of thumb, the number of classes $k = \sqrt{50} \approx 7$. Due to the resolution of the measurement values of 0.1 mm here, the "more precise" giving of the class limit by one decimal position is presented.

Class	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Lower class limit	*5.45*	*5.65*	*5.85*	*6.05*	*6.25*	*6.45*	*6.65*
Upper class limit	*5.65*	*5.85*	*6.05*	*6.25*	*6.45*	*6.65*	*6.85*

Class	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Absolute frequency	*3*	*5*	*10*	*14*	*9*	*7*	*2*
Relative frequency	*6 %*	*10 %*	*20 %*	*28 %*	*18 %*	*14 %*	*4 %*

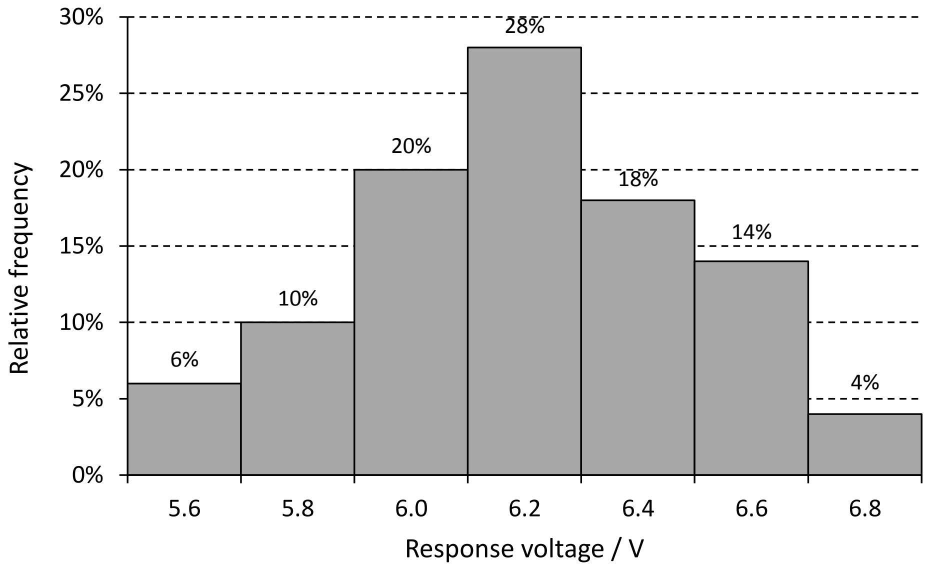

Fig. 3: Histogram of the above data

The Normal Probability Plot

The Gaussian bell-shaped curve is a representation of the probability density function of the normal distribution. The distribution function of the normal distribution is produced by the integral over the density function. Their values are listed in Table 4. The graphic representation of the distribution function $\Phi(u)$ is an s-shaped curve.

If one distorts the y-axis of this representation in a manner that the s-shaped curve becomes a straight line, a new coordinate system is produced, the probability plot. The x-axis remains unchanged. Due to this correlation, a representation of a normal distribution is always produced as a straight line in such a probability plot.

This fact can be made use of in order to graphically test a given data set for normal distribution. As long as the number of given values is large enough, a histogram of these values is created and the relative frequencies of values within the classes of a classification are determined. If the relative cumulative frequencies found are plotted over the right class limit and a sequence of dots is thereby produced which lie approximately in a straight line, then it can be concluded that the values of the data set are approximately normally distributed.

EXAMPLE:

The "response voltages of relays" example is assumed from the previous section "Histogram":

Class	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Lower class limit	*5.45*	*5.65*	*5.85*	*6.05*	*6.25*	*6.45*	*6.65*
Upper class limit	*5.65*	*5.85*	*6.05*	*6.25*	*6.45*	*6.65*	*6.85*
Absolute frequency	*3*	*5*	*10*	*14*	*9*	*7*	*2*
Relative frequency	*6 %*	*10 %*	*20 %*	*28 %*	*18 %*	*14 %*	*4 %*
Absolute cumulative frequency	*3*	*8*	*18*	*32*	*41*	*48*	*50*
Relative cumulative frequency	*6 %*	*16 %*	*36 %*	*64 %*	*82 %*	*96 %*	*100 %*

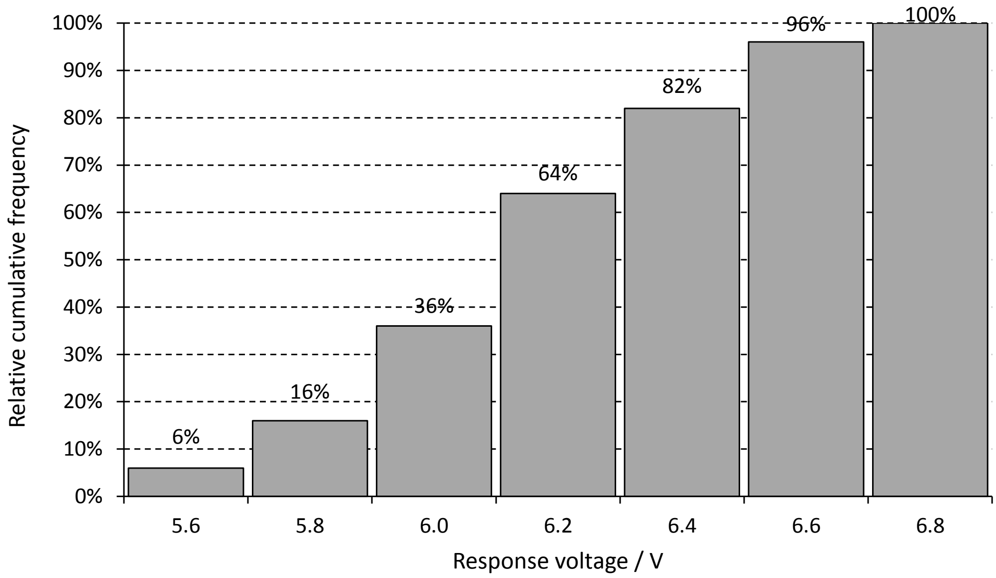

Fig. 4

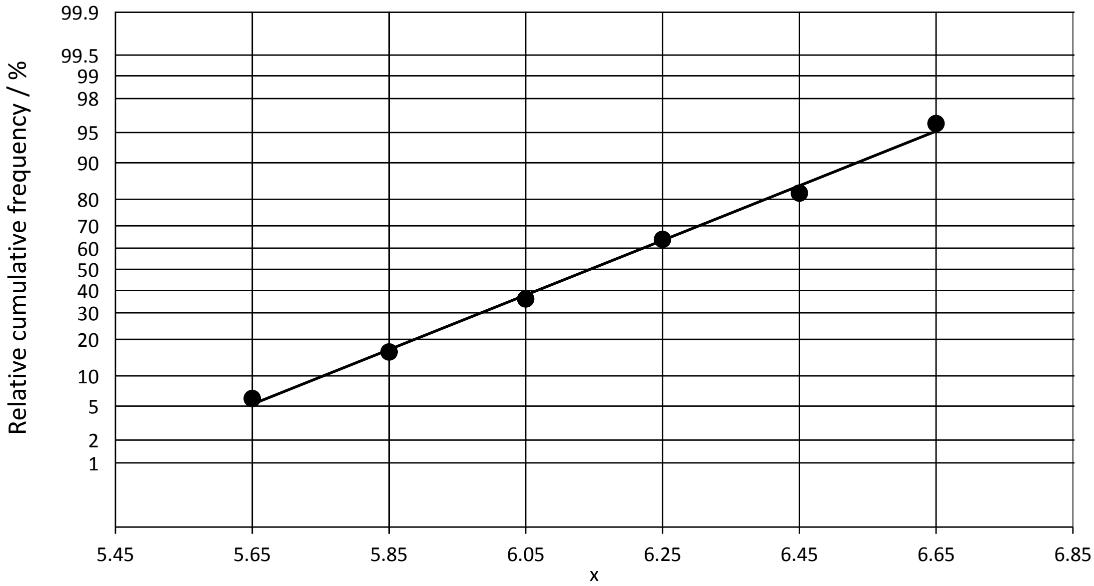

Fig. 5: Representation of the measurement values in form of a probability plot:
x-axis: Scaling corresponding to the class limits

Representation of Samples with Small Sizes in a Probability Plot

Problem:

Values x_i of a data set are present, whose number n, however, is not sufficient for the creation of a histogram. The individual values x_i should have cumulative frequencies allocated to them so that creation of a probability plot is possible.

Input quantities:

x_i, n

Procedure:

1. The values $x_1, x_2, \ldots, x_n$ are arranged in order of magnitude:

   $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$.

   The smallest value $x_{(1)}$ has the rank 1, and the largest value $x_{(n)}$ has the rank n.

2. Each $x_{(i)}$ $(i = 1, 2, \ldots, n)$ has a cumulative frequency $H_i(n)$ allocated to it according to Table 1:

   $x_{(1)}$, $x_{(2)}$, ... , $x_{(n)}$

   $H_1(n)$, $H_2(n)$, ... , $H_n(n)$.

3. Representation of the points ($x_{(i)}$, $H_i(n)$) in a probability plot.

Remark:

The cumulative frequency $H_i(n)$ for rank number i can also be calculated with one of the approximation formulas

$$H_i(n) = \frac{i-0.5}{n} \qquad \text{and} \qquad H_i(n) = \frac{i-0.3}{n+0.4}.$$

The deviation from exact table values is insignificant here.

EXAMPLE:

The following sample of 10 measurement values should be graphically tested for normal distribution:

2.1	*2.9*	*2.4*	*2.5*	*2.5*	*2.8*	*1.9*	*2.7*	*2.7*	*2.3.*

The values are arranged in order of magnitude:

1.9	*2.1*	*2.3*	*2.4*	*2.5*	*2.5*	*2.7*	*2.7*	*2.8*	*2.9.*

The value 1.9 has the rank 1, the value 2.9 has the rank 10. In Table 1 in the appendix (sample size $n = 10$*), the cumulative frequencies (in percent) can be found for each rank number* i *:*

6.2	*15.9*	*25.5*	*35.2*	*45.2*	*54.8*	*64.8*	*74.5*	*84.1*	*93.8.*

Afterwards, a suitable partitioning (scaling) is selected for the x-axis of the probability plot corresponding to the values 1.9 to 2.9 and the cumulative frequencies are recorded over the corresponding well-ordered sample values. In the example considered, then, the following points are marked:

(1.9, 6.2), (2.1, 15.9), (2.3, 25.5), . . . , (2.7, 74.5), (2.8, 84.1), (2.9, 93.8).

Since these points can be approximated quite well by a straight line, one can assume that the sample values are approximately normally distributed.

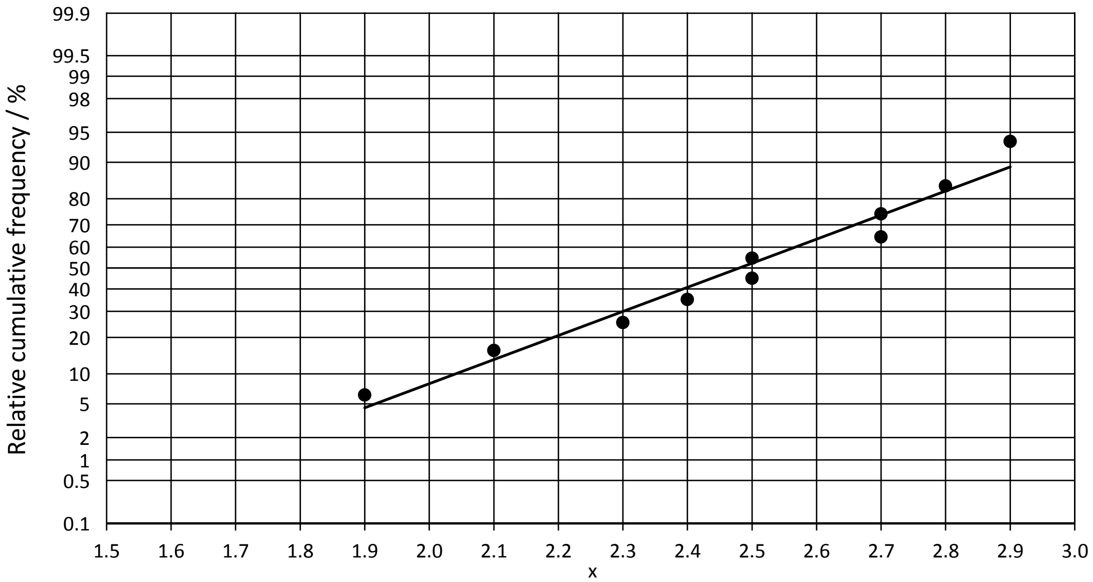

Fig. 6: Representation of the measurement values in a probability plot:
x-axis: scaling corresponding to the measurement values

Test of Normality

Problem:

It should be inspected by way of a simple test whether the n values of a sample can originate from a normal distribution.

Input quantities:

x_i, s, n

Procedure:

1. Determine the smallest value x_{min} and the largest value x_{max} of the sample and the difference between these values:

   $R = x_{max} - x_{min}$ (range)

2. Calculate the number

   $$Q = \frac{R}{s}$$

3. Find the lower limit LL and the upper limit UL for Q for sample size n and the desired (chosen) level of significance α (e.g. α = 0.5 %) in Table 2 and compare Q with these limits.

Test outcome:

If Q lies outside of the interval given by LL and UL, then

$Q \le LL$ or $Q \ge UL$ applies,

then the sample cannot originate from a normal distribution (level of significance α).

> *REMARK:*
> *If Q exceeds the upper limit, this can also be caused by an outlier (x_{max}) (compare with outlier test).*

EXAMPLE:

During a receiving inspection, a sample of 40 parts was taken from a delivery and was measured. It should be inspected through a simple test as to whether or not the following characteristic values of the parts taken can originate from a normal distribution.

29.1	25.1	27.1	25.1	29.1	27.7	26.1	24.9
29.7	26.4	26.8	28.5	26.9	29.8	28.7	26.6
26.4	25.8	29.3	27.4	25.1	27.6	30.0	27.7
26.0	28.9	26.6	29.5	25.6	27.4	25.7	28.3
29.8	24.5	27.0	27.4	26.2	28.6	25.1	24.6

$x_{max} = 30.0 \qquad x_{min} = 24.5 \qquad \bar{x} = 27.2 \qquad s = 1.641$

$R = x_{max} - x_{min} = 30.0 - 24.5 = 5.5$

$Q = \frac{R}{s} = \frac{5.5}{1.641} = 3.35$ *Lower limit* $LL = 3.41$ *for* $n = 40$ *and* $\alpha = 0.5\%$

$Q = 3.35 < 3.41 = LL$

The range of the sample values is too small in relation to the standard deviation. With a level of significance of 0.5 % the sample can not originate from a normally distributed population.

It can be suspected from a histogram of the sample values that the delivery was 100 % sorted at the supplier.

REMARK:

This simple test inspects merely the relationship of range and standard deviation which lies within the interval $[LL, UL]$ *dependent on* n *and* α *with normally distributed values. No comparison occurs, for example, between calculated relative cumulative frequencies and the values of the theoretical distribution function of the normal distribution.*

In particular, in the case $LL < Q < UL$*, the conclusion cannot be drawn that normally distributed measurement data are actually present.*

Outlier Test

Problem:

A measurement series of n values is present from which can be assumed with quite great certainty that it originates from a normal distribution.

This assumption is based in a subjectively logical manner, for example, on a longer-term observation of the measurement object or is justified through the evaluation of a histogram or a representation of the values in a probability plot. It should be decided as to whether a value of the measurement series which is unusually large (x_{max}) or unusually small (x_{min}) may be handled as an outlier.

Input quantities:

x_{max}, x_{min}, $\bar{x}$, s, n

Procedure:

1. Calculate the difference

   $$R = x_{max} - x_{min}$$

2. Calculate the number

   $$Q = \frac{R}{s}$$

3. Find the upper limit UL (for Q) for the size of the measurement series (sample) n and the desired level of significance α (e.g. α = 0.5 %) in Table 3. Compare Q with this upper limit.

Test outcome:

If Q exceeds the upper limit UL, i.e. $Q \geq UL$, one of the two values x_{max} and x_{min} which is farthest away from the mean $\bar{x}$ can be considered as an outlier.

EXAMPLE:

Within the scope of a machine capability analysis, the following characteristic values were measured with 50 parts produced one after the other:

68.4	69.6	66.5	70.3	70.8	66.5	70.7	67.6	67.9	63.0
66.8	65.3	70.2	74.1	66.9	65.4	64.4	66.1	67.2	69.5
67.9	64.2	67.3	66.2	61.7	64.3	61.8	63.1	62.4	68.3
67.3	62.5	65.3	68.0	67.4	66.7	86.0	67.1	69.8	65.3
73.0	70.9	67.3	67.9	67.4	65.1	71.2	62.0	67.5	67.4

The graphical recording of these measurement values enables the recognition that the value 86.0 is comparably far away from the mean in comparison to all other values. In the calculation of the mean (broken line), the possible outlier was not considered. A normal probability plot of the remaining values indicates that the values are approximately normally distributed.

It should be explained whether all values can originate from the same population or whether a real outlier is being dealt with with the extreme value.

$x_{max} = 86.0 \qquad x_{min} = 61.7 \qquad \bar{x} = 67.01 \qquad s = 3.894$

$R = x_{max} - x_{min} = 86.0 - 61.7 = 24.3$

$Q = \frac{R}{s} = \frac{24.3}{3.894} = 6.24$ *Upper limit* $UL = 5.91$ *for* $n = 50$ *and* $\alpha = 0.5\,\%$

$Q = 6.24 > 5.91 = OS$

A test of the components belonging to the outlier presented the result that an unworked piece from another material batch was accidentally processed during this manufacture.

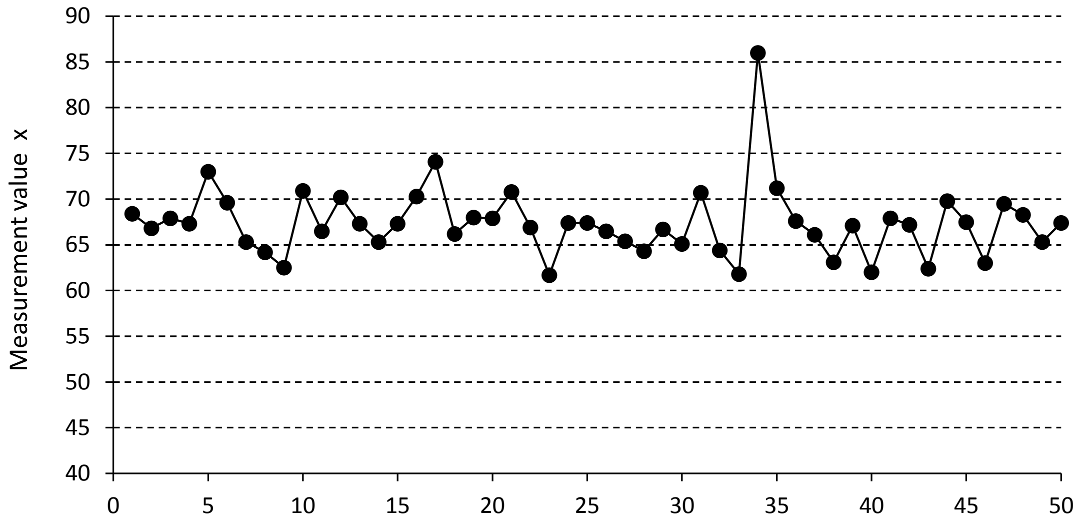

Fig. 7: Representation of the measurement values of the characteristic X in the form of a dot sequence

Normal Distribution, Estimation of Fractions Nonconforming

Problem:

A sample of n values x_i of a normally distributed quantity x, e.g. a characteristic of a product is present. A tolerance interval is given for the characteristic which is limited by the lower specification limit LSL and the upper specification limit USL.

On the basis of a sample, the portions of the characteristic values are to be estimated which lie a) below LSL, b) between LSL and USL, c) above USL.

Input quantities:

$\bar{x}$ and s of the sample, LSL, USL

Procedure:

1. Calculate

$$u_1 = \frac{\bar{x} - UGW}{s} \qquad \text{and} \qquad u_2 = \frac{OGW - \bar{x}}{s}$$

2. Determine the values $\Phi(u_1)$ and $\Phi(u_2)$ from the table of the standard normal distribution (Table 4)

Result:

a) Below LSL lie $\Phi(u_1) \cdot 100\%$

b) Between LSL and USL lie $(\Phi(u_2) - \Phi(u_1)) \cdot 100\%$

c) Above USL lie $(1 - \Phi(u_2)) \cdot 100\%$

of all values.

REMARK:

The estimated values calculated in this manner are full of uncertainty which becomes greater the smaller the sample size n *is and the more the distribution of the characteristic deviates from the normal distribution.*

If fractions nonconforming are to be given in ppm (1 ppm = 1 part per million), the factor 100 % is to be replaced by 1,000,000 ppm in the calculation of the outcome.

EXAMPLE:

$\bar{x} = 64.15\,mm \quad s = 0.5\,mm \quad LSL = 63.1\,mm \quad USL = 65.0\,mm$

$$u_1 = \frac{63.1\,mm - 64.15\,mm}{0.5\,mm} = -2.1 \qquad u_2 = \frac{65.0\,mm - 64.15\,mm}{0.5\,mm} = 1.7$$

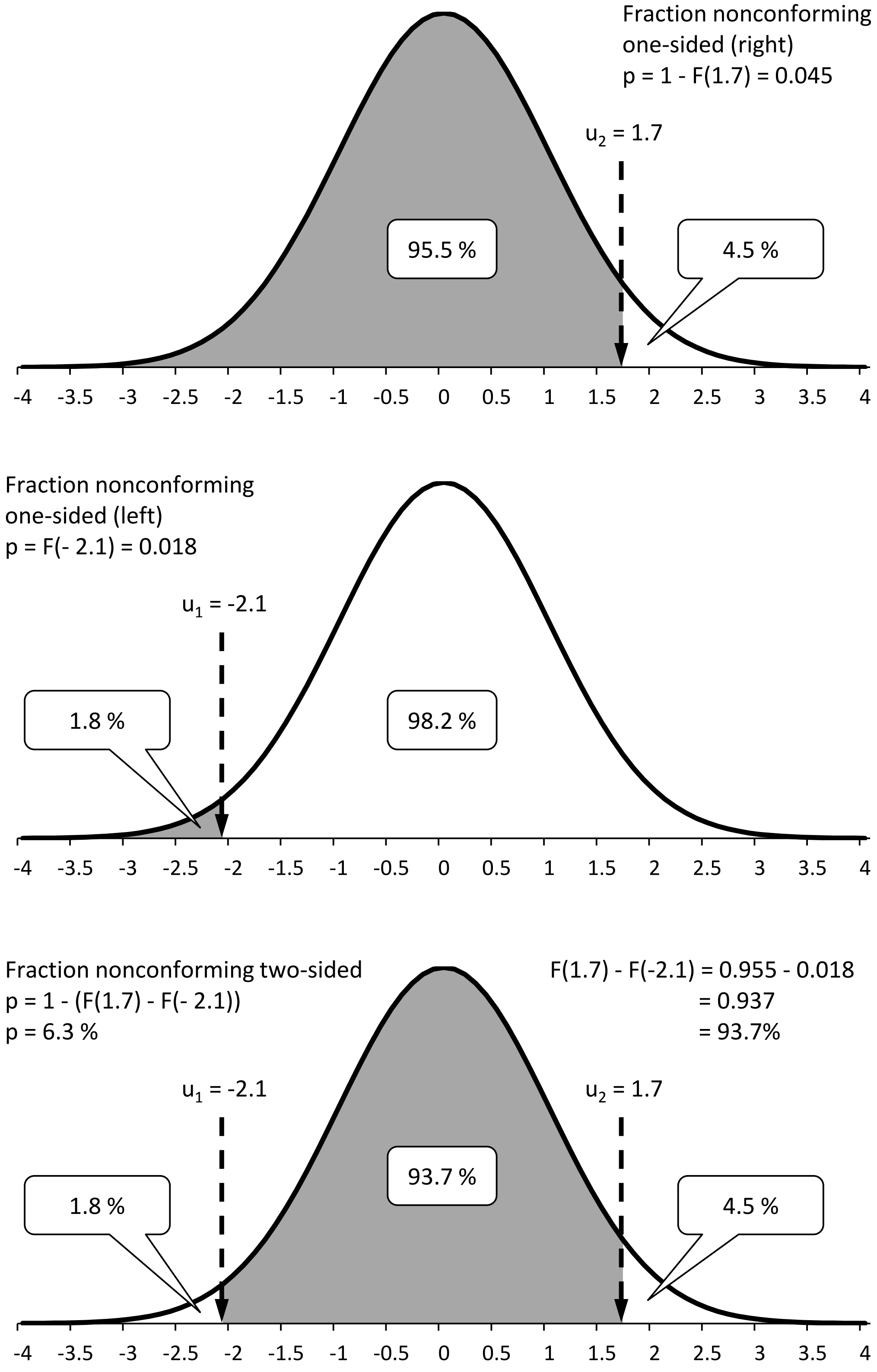

Fig. 8

Normal Distribution, Estimation of Confidence Limits

Problem:

A sample of n values x_i of a normally distributed quantity x is present, e.g. a characteristic of a product. The limits LSL and USL of a region symmetric to $\bar{x}$ are to be given in which the portion p (e.g. 99 %) of the population of all characteristic values upon which the sample is based lies with the probability P (e.g. 95 %).

Input quantities:

$\bar{x}$ and s of the sample, P, p, n

Procedure:

1. Calculate the number

   $$\Phi = \frac{1+p}{2}$$

2. Find the number u for value Φ in the table of the standard normal distribution (Table 4).
3. Find the factor k for the number n of the values and the probability P (Table 5).
4. Calculate

   $LSL = \bar{x} - k \cdot u \cdot s$ and

   $USL = \bar{x} + k \cdot u \cdot s$

Result:

The portion p of all characteristic values lies between the limits $\bar{x} - k \cdot u \cdot s$ and $\bar{x} + k \cdot u \cdot s$ with the probability P.

EXAMPLE: Estimation of confidence limits

The following injection volumes were measured during experiments on a test bench on 25 injection valves (mass per 1000 strokes in g):

Injection volumes (mass per 1000 strokes in g)									
7.60	7.64	7.66	7.71	7.66	7.52	7.70	7.56	7.66	7.60
7.60	7.64	7.63	7.65	7.59	7.59	7.55	7.62	7.67	7.69
7.62	7.70	7.60	7.64	7.71					

A normal probability plot of this measurement outcome indicates that a normal distribution can be assumed. The development department would like to fix a tolerance interval [LSL,USL] symmetric to the mean in which $p = 95\,\%$ of all injection volumes of the total population lie with a probability of 99 % ($P = 99\,\%$).

$\bar{x} = 7.6324 \qquad s = 0.0506 \qquad n = 25$

$$p = 95\,\% \quad \Rightarrow \quad \Phi = \frac{1+p}{2} = \frac{1.95}{2} = 0.975$$

$u(0.975) = 1.96 \qquad k\,(n = 25, P = 99\,\%) = 1.52$

$LSL = 7.6324 - 1.52 \cdot 1.96 \cdot 0.0506 \approx 7.48$

$USL = 7.6324 + 1.52 \cdot 1.96 \cdot 0.0506 \approx 7.78$

Confidence Interval for the Mean

Problem:

The characteristic quantities mean $\bar{x}$ and standard deviation s calculated from n sample values represent merely estimations for the generally unknown characteristic quantities μ and σ of the population of all values upon which the sample is based. A region should be given around $\bar{x}$ in which μ lies with great probability P (e.g. 99 %).

Input quantities:

$\bar{x}$, s, n, P

Procedure:

1. Find the quantity t for the degrees of freedom $f = n - 1$ and the desired probability P (e.g. 99 %) in Table 6.
2. Calculate

   the lower interval limit $\bar{x} - t \cdot \frac{s}{\sqrt{n}}$ and the

   upper interval limit $\bar{x} + t \cdot \frac{s}{\sqrt{n}}$.

Result:

The unknown mean μ lies between the calculated interval limits with the probability P (e.g. 99 %), i.e. the following applies:

$$\bar{x} - t \cdot \frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t \cdot \frac{s}{\sqrt{n}} .$$

EXAMPLE:

Within the scope of a process capability analysis, the mean $\bar{x} = 74.51$ and the standard deviation $s = 1.38$ were determined from the $n = 125$ individual values of a completely filled-out quality control chart ($\bar{x}$-s-chart). Corresponding with the outcome of the stability test, the process possesses a stable average.

For $P = 99\,\%$ and $f = 125 - 1 = 124$, one finds the value $t = 2.63$ in Table 6 (this value belongs to the next smaller degree of freedom $f = 100$ listed in the table).

Use of both formulas produces:

lower interval limit: $74.51 - 2.63 \cdot \frac{1.38}{\sqrt{125}} = 74.18$

upper interval limit: $74.51 + 2.63 \cdot \frac{1.38}{\sqrt{125}} = 74.83$.

The unknown process average μ lies with a probability of 99 % within the interval [74.18; 74.83].

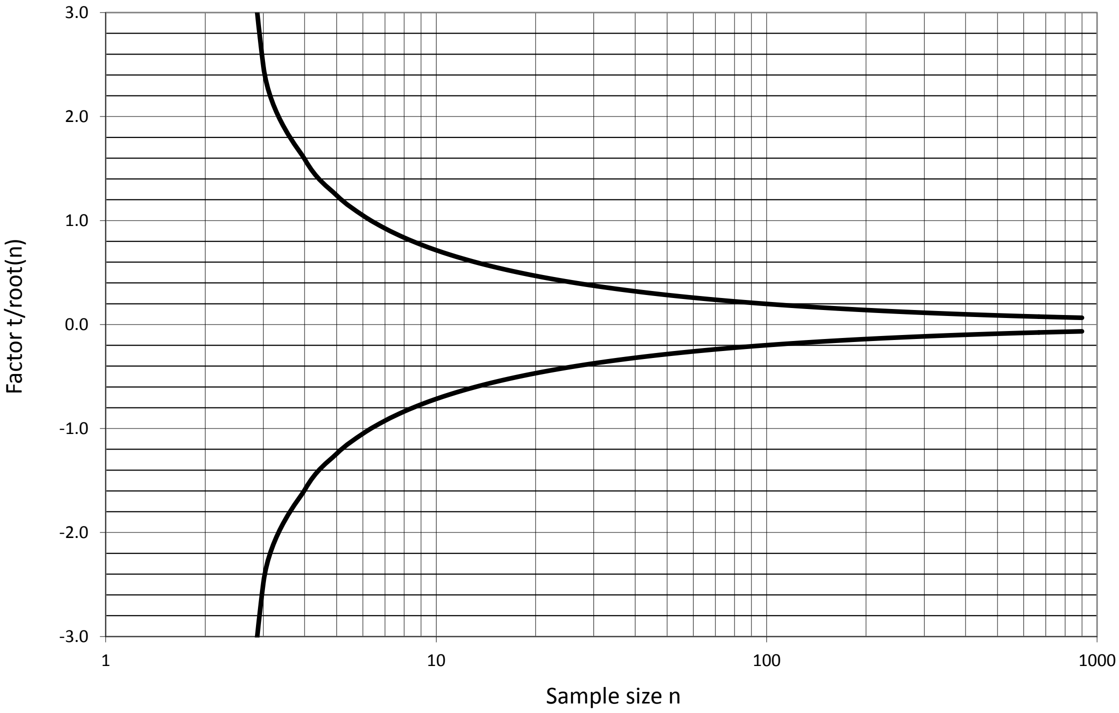

Fig. 9: The representation illustrates the dependence of the confidence interval of the mean μ on the sample size n .

Confidence Interval for the Standard Deviation

Problem:

The variance s^2 calculated from n sample values is an estimate of the unknown variance σ^2 of the population of all values upon which the sample is based. A confidence interval for the standard deviation σ should be given; an interval, then, around s in which σ lies with great probability P (e.g. 99 %).

Input quantities:

s, n, P

Procedure:

1. Find the quantities c_1 and c_2 for sample size n and the selected probability P (e.g. 99 %) in Table 7.
2. Calculate

   the lower interval limit $c_1 \cdot s$ and the

   upper interval limit $c_2 \cdot s$.

Result:

The unknown standard deviation σ lies between the calculated interval limits with the probability P (e.g. 99 %). Applicable, then, is:

$c_1 \cdot s \leq \sigma \leq c_2 \cdot s$.

EXAMPLE:

A measuring instrument capability analysis includes the determination of the repeatability standard deviation s_W. *For this, at least 25 measurements of a standard are conducted and evaluated at one standard.*

Within the scope of such a test on an inductivity measurement device, the standard deviation $s = s_W = 2.77$ *was calculated from* $n = 25$ *measurement outcomes. The measurement outcomes are approximately normally distributed.*

For sample size $n = 25$ *and probability* $P = 99\,\%$ *one finds the values* $c_1 = 0.72$ *and* $c_2 = 1.56$ *in Table 7. Insertion produces:*

lower interval limit: $c_1 \cdot s = 0.72 \cdot 2.77 = 1.99$

upper interval limit: $c_2 \cdot s = 1.56 \cdot 2.77 = 4.32$.

The unknown standard deviation σ *which characterizes the variability of the measurement device lies within the interval [1.99; 4.32] with a probability of 99 %.*

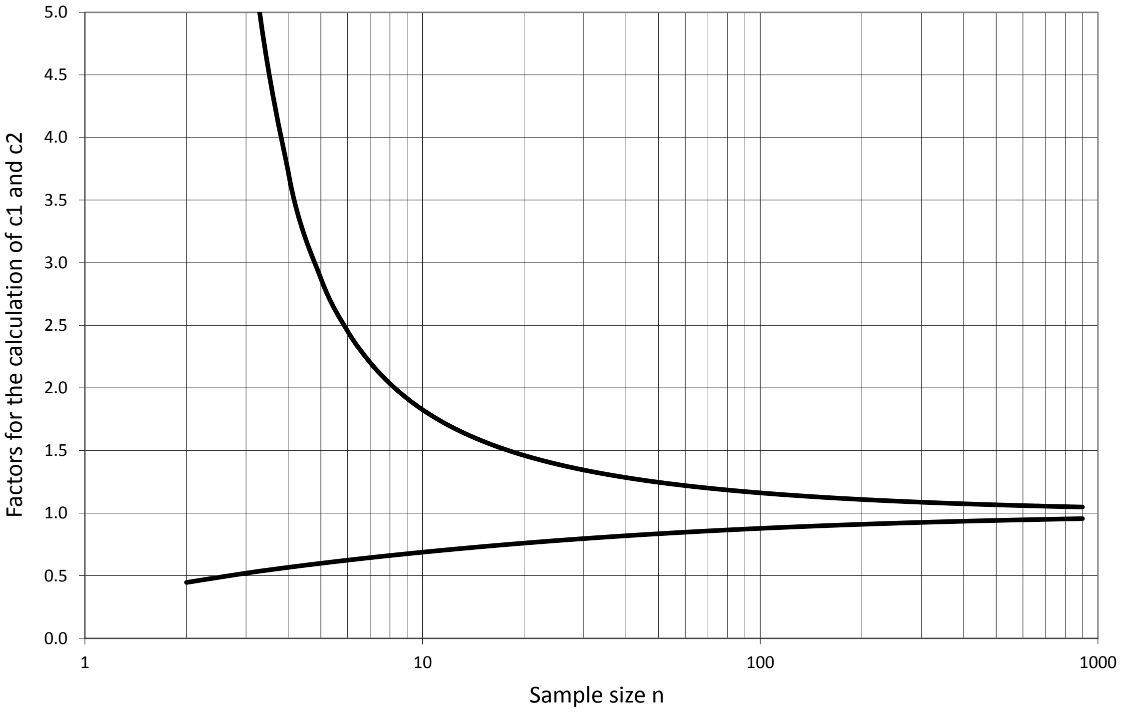

Fig. 10: The representation illustrates the dependency of the confidence interval of the standard deviation σ on the sample size n .

Comparison of Two Variances (F Test)

Problem:

It should be decided whether or not the variances s_1^2 and s_2^2 of two data sets of size n_1 and n_2 are significantly different.

Input quantities:

s_1^2, s_2^2, n_1, n_2

Procedure:

1. Calculate the test statistic F

   $F = \frac{s_1^2}{s_2^2}$ s_1^2 is the greater of the two variances and is located below the fraction line.

   n_1 is the sample size belonging to s_1^2.

2. Compare the test statistic F with the table values $F(95\%)$ and $F(99\%)$ to the degrees of freedom $f_1 = n_1 - 1$ and $f_2 = n_2 - 1$ (Table 9).

Test outcome:

If ...	... the statistical difference between both variances is
$F \geq F(99\%)$	highly significant
$F(99\%) > F \geq F(95\%)$	significant
$F(95\%) > F$	insignificant

EXAMPLE: Comparison of two variances

By using two different granular materials A and B, 10 injection molding parts were respectively manufactured and measured. Both measurement series which give the relative shrinkage of the parts should be compared with respect to their variances.

	Measurement outcomes (relative shrinkage)									
Gran. mat. A	*0.16*	*0.30*	*0.26*	*0.24*	*0.33*	*0.28*	*0.24*	*0.18*	*0.35*	*0.30*
Gran. mat. B	*0.32*	*0.26*	*0.36*	*0.22*	*0.14*	*0.23*	*0.40*	*0.19*	*0.32*	*0.12*

	Evaluation		
	$\bar{x}$	s	s^2
Gran. mat. A	*0.264*	*0.06*	*0.0037*
Gran. mat. B	*0.256*	*0.093*	*0.0087*

F test: $F = \frac{0.0087}{0.0037} = 2.35$ *Degrees of freedom:* $f_1 = 9$, $f_2 = 9$

Table values: $F(95\%) = 4.03$ $F(99\%) = 6.54$ *(Table 9)*

Test decision:

Because $F < F(95\%)$*, the quantitative difference between both variances is insignificant.*

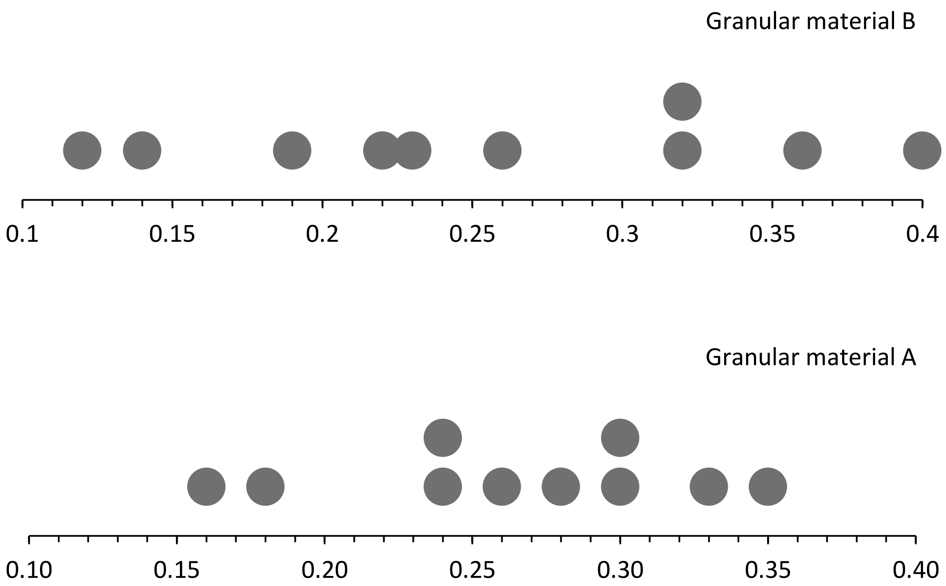

Fig. 11: Dot diagrams for the measurement outcome

Comparison of Two Means (Equal Variances)

Problem:

It should be decided whether the means $\bar{x}_1$ and $\bar{x}_2$ of two data sets of size n_1 and n_2 are significantly different or whether both data sets originate from a mutual population.

Input quantities:

$\bar{x}_1,\ \bar{x}_2,\ s_1^2,\ s_2^2,\ n_1,\ n_2$

Prerequisite:

It was indicated by an F test that the variances s_1^2 and s_2^2 are not significantly different.

Procedure:

1. Calculate the test characteristic

$$t = \sqrt{\frac{n_1 \cdot n_2 \cdot (n_1 + n_2 - 2)}{n_1 + n_2}} \cdot \frac{\left|\bar{x}_1 - \bar{x}_2\right|}{\sqrt{(n_1 - 1)\cdot s_1^2 + (n_2 - 1)\cdot s_2^2}} \qquad \text{for } n_1 \neq n_2$$

and

$$t = \sqrt{n} \cdot \frac{\left|\bar{x}_1 - \bar{x}_2\right|}{\sqrt{s_1^2 + s_2^2}} \qquad \text{for } n_1 = n_2 = n\,.$$

2. Compare the test characteristic t with the table values $t(95\%)$ and $t(99\%)$ to the degrees of freedom $f = n_1 + n_2 - 2$ (Table 6).

Test outcome:

If ...	... the statistical difference between both means is
$t \geq t(99\%)$	highly significant
$t(99\%) > t \geq t(95\%)$	significant
$t(95\%) > t$	insignificant

EXAMPLE: t Test

A new welding process is to be introduced in a welding procedure. The tensile strength of the weld joint was measured on 10 respective parts which were processed with the old process and with the new process.

On the basis of the measured values (force in kN), it should be decided whether the new procedure produces tensile strengths which are significantly better.

	Measurement outcomes											$\bar{x}$	s	s^2
Old process	2.6	2.0	1.9	1.7	2.1	2.2	1.4	2.4	2.0	1.6		1.99	0.36	0.13
New process	2.1	2.9	2.4	2.5	2.5	2.8	1.9	2.7	2.7	2.3		2.48	0.31	0.1

F test: $F = \frac{0.13}{0.1} = 1.3$ *Degrees of freedom:* $f_1 = 9,\; f_2 = 9$

Table values: $F(95\%) = 4.03$ $F(99\%) = 6.54$

The variances of the measurement outcomes are not significantly different and the prerequisite for the t test is thereby fulfilled.

$$t = \sqrt{10} \cdot \frac{|1.99 - 2.48|}{\sqrt{0.13 + 0.1}} = \sqrt{10} \cdot \frac{0.49}{0.48} = 3.2$$

$n_1 = n_2 = 10$ *Degrees of freedom:* $f = 10 + 10 - 2 = 18$

Table value: $t(95\%) = 2.10$ $t(99\%) = 2.88$ *(Table 6)*

$t = 3.2 > 2.88 = t(99\%)$

The difference in the means of the two measurement series is highly significant.

Since the mean $\bar{x}_2$ *is greater than* $\bar{x}_1$*, the tensile strength increased, then, and the new procedure should be introduced.*

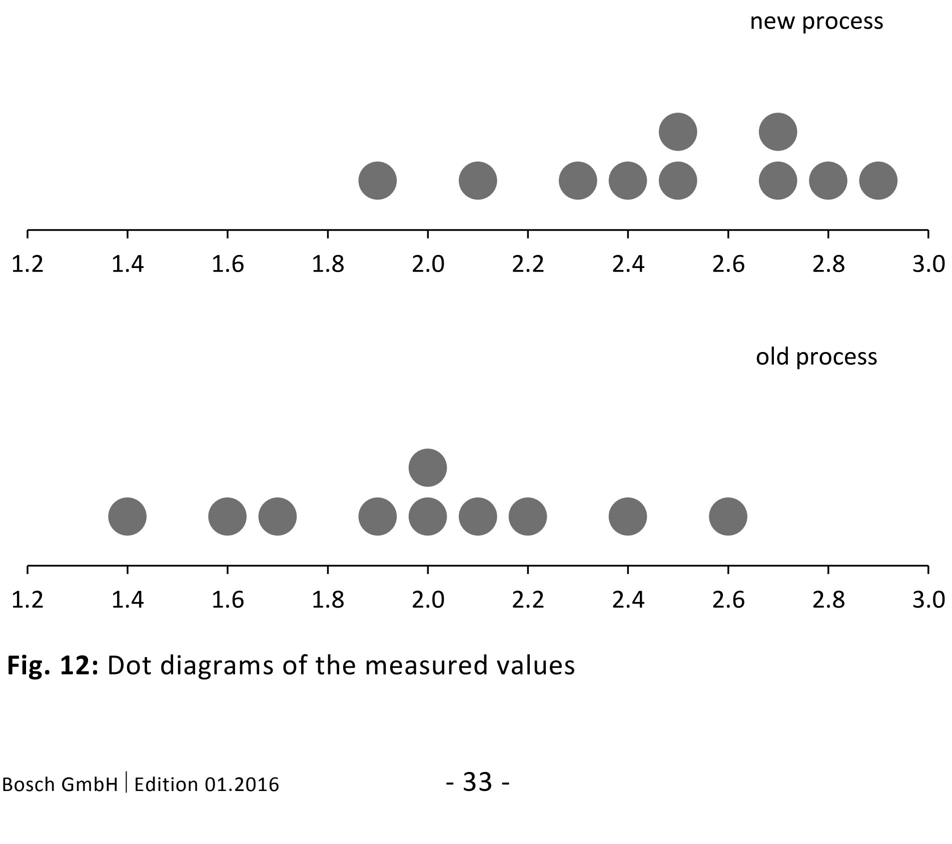

Fig. 12: Dot diagrams of the measured values

Comparison of Two Means (Unequal Variances)

Problem:

It should be decided whether the means $\bar{x}_1$ and $\bar{x}_2$ of two data sets are significantly different or whether the two data sets originate from a mutual population.

Input quantities:

$\bar{x}_1,\ \bar{x}_2,\ s_1^2,\ s_2^2,\ n_1,\ n_2$

Prerequisite:

It was indicated by an F test that the difference between s_1^2 and s_2^2 is not coincidental.

Procedure:

1. Calculate the test characteristic t : $$t = \frac{\left|\bar{x}_1 - \bar{x}_2\right|}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}.$$

2. Calculate the degrees of freedom f : $$f = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2 - 1}}.$$

3. Compare the test characteristic t with the table values $t(95\%)$ and $t(99\%)$ (Table 6) to the degrees of freedom f (round f to a whole number!).

Test outcome:

If ...	... the statistical difference between both means is
$t \geq t(99\%)$	highly significant
$t(99\%) > t \geq t(95\%)$	significant
$t(95\%) > t$	insignificant

EXAMPLE:

An adjustment standard was measured many times with two different measurement procedures A and B.

With procedure A, $n_1 = 8$, and with procedure B $n_2 = 16$ measurements were conducted. The following deviations from the nominal value of the adjustment standard were determined:

Measured values (procedure A)							
4.4	4.1	1.0	3.8	2.3	4.4	6.3	2.9

Measured values (procedure B)							
6.9	7.6	7.6	8.2	8.2	8.1	8.5	7.9
7.2	8.0	9.3	8.1	7.1	7.7	7.3	6.9

It should be decided whether the means of both data sets are significantly different or not. Significantly different means mean in this case that both measurement procedures have significantly different systematic errors.

	Evaluation		
	n_i	$\bar{x}_i$	s_i^2
Process A (Index 1)	8	3.65	2.54
Process B (Index 2)	16	7.7875	0.4065

F test: $F = \frac{2.54}{0.4065} = 6.25 > 4.85 = F_{7;15;0.99}$ *(Table 9)*

The difference between the two variances is not coincidental.

Test characteristic: $t = \frac{|3.65 - 7.7875|}{\sqrt{\frac{2.54}{8} + \frac{4.065}{16}}} = 7.07$

Degrees of freedom: $f = \frac{\left(\frac{2.54}{8} + \frac{4.065}{16}\right)^2}{\frac{\left(\frac{2.54}{8}\right)^2}{8-1} + \frac{\left(\frac{4.065}{16}\right)^2}{16-1}} \approx 8$

Comparison with the table value: $t = 7.07 > 3.355 = t(99\%)$ *(Table 6)*

Test outcome:

The difference between the means is highly significant.

Minimum Sample Size

With the assistance of a t test it can be decided whether the mean of two samples (data sets, measurement series) are significantly different. This decision becomes more certain the greater the number of available measurement values.

Since, though, expense and costs increase with the number of measurements (experiments), an experimenter should, for example, already consider during the preparation for the experiment which minimum mean difference is of interest to him and which minimum sample size n he must choose, so that this minimum difference of means will be recognizable (significant) on the basis of the evaluation of the experiment.

It is here necessary to give at least a rough estimate of the standard deviation s of the measurement values to be expected and to express the interested difference of the means as a multiple of s.

The following rule of thumb is used in practice.

Mean interval which should be recognized:	Necessary minimum number n of values per measurement series:
$3 \cdot s$	$n \geq 6$
$2 \cdot s$	$n \geq 15$
$1 \cdot s$	$n \geq 30$

REMARK:

In the comparison of means of two measurement series and the corresponding test decision, two types of error are possible.

In the first case, both measurement series originate from the same population, i.e. there is no significant difference. If one decides on the basis of a t test that a difference exists between the two means, one is making a type 1 error, "α". It corresponds to the level of significance of the t test (e.g. $\alpha = 1\%$).

If, in the second case, an actual difference of the means exists, i.e. the measurement series originate from two different populations, this is not indicated with absolute certainty by the test. The test outcome can coincidentally indicate that this difference does not exist. One refers in this case to a type 2 error, "β".

Both errors can be uncomfortable for an experimenter, since he may possibly suggest expensive continued examinations or even modifications in a production procedure on the basis of the significant effects of a factor (type 1 error), or because he really doesn't recognize the significant effects present and the change for possible procedure improvements escapes him (type 2 error).

The minimum sample size n, which is necessary in order to recognize an actual mean difference, is dependent corresponding to the above plausibility consideration of the interval of the two means and the levels of significance α and β given in units of the standard deviation σ.

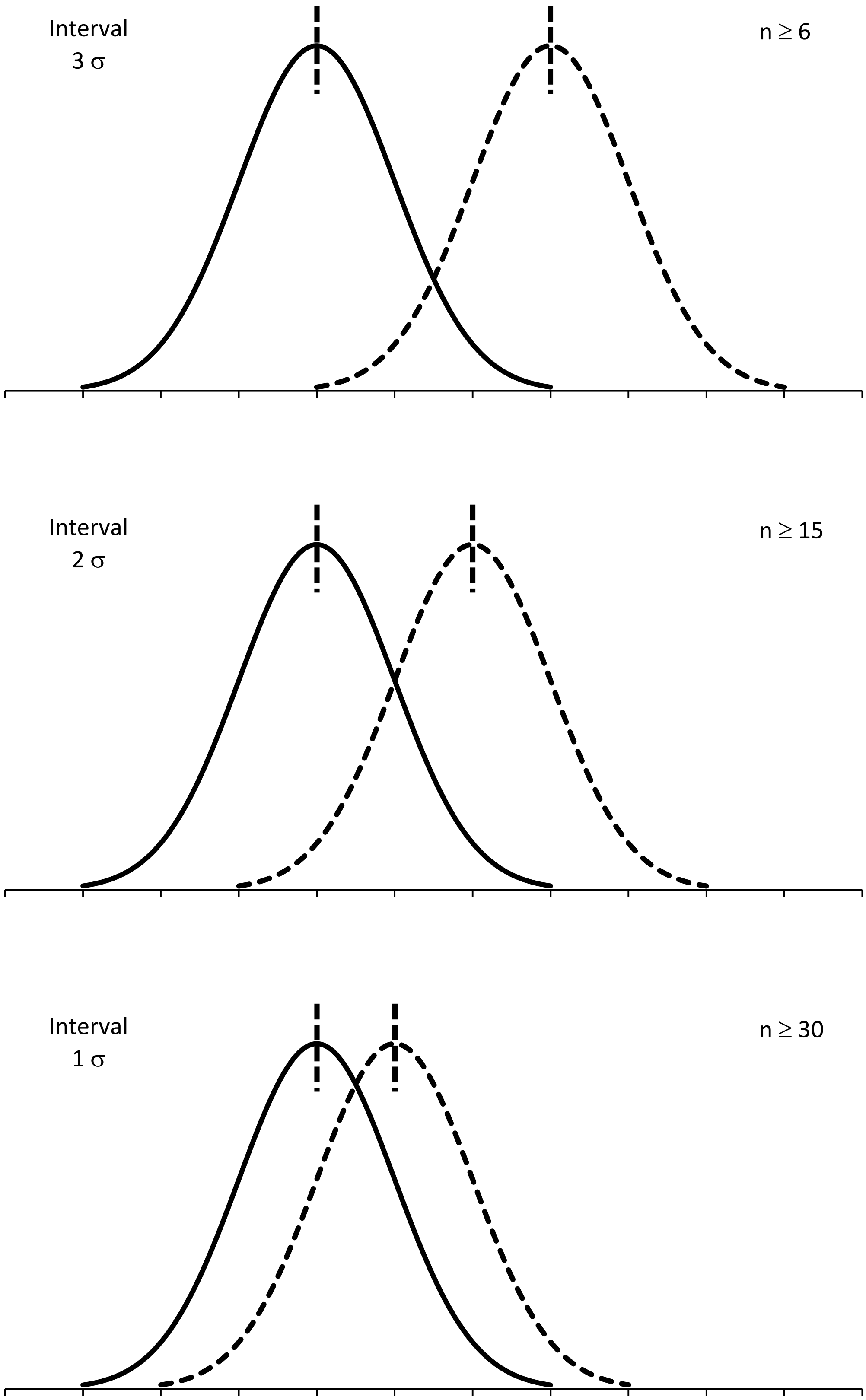

Fig. 13: Schematic explanation of the minimum sample size

Comparison of Several Means

Problem:

The outcome of an experimental investigation with k individual experiments, which were repeated respectively n times, is represented by k means $\bar{y}_i$ and k variances s_i^2:

Experiment No.	Outcomes	Mean	Variance
1	$y_{11}, y_{12}, \ldots, y_{1n}$	$\bar{y}_1$	s_1^2
2	$y_{21}, y_{22}, \ldots, y_{2n}$	$\bar{y}_2$	s_2^2
.	.	.	.
.	.	.	.
.	.	.	.
k	$y_{k1}, y_{k2}, \ldots, y_{kn}$	$\bar{y}_k$	s_k^2

It should be decided whether the means $\bar{y}_i$ of k experiments are significantly different, or whether the differences are only coincidental. This procedure is called one-way ANOVA (analysis of variance) and is usually used in the first evaluation of experiments on the basis of statistical experimental designs (compare with factorial variance analysis).

Input quantities:

y_i, s_i^2, n, k

Process:

1. Calculate the mean variance

$$\overline{s^2} = \frac{1}{k} \cdot \sum_{i=1}^{k} s_i^2 \,.$$

2. Calculate the variance of the means

$$s_{\bar{y}}^2 = \frac{1}{k-1} \cdot \sum_{i=1}^{k} (\bar{y}_i - \bar{\bar{y}})^2 \qquad \text{with} \qquad \bar{\bar{y}} = \frac{1}{k} \cdot \sum_{i=1}^{k} \bar{y}_i \qquad \text{(Total mean).}$$

3. Compare the test statistic

$$F = \frac{n \cdot s_{\bar{y}}^2}{\overline{s_y^2}}$$

with the table values F(95%) and F(99%) to the degrees of freedom $f_1 = k-1$ and $f_2 = (n-1) \cdot k$ (Table 8).

Test outcome:

If ...	... the statistical difference of both variances is
$F \geq F(99\%)$	highly significant
$F(99\%) > F \geq F(95\%)$	significant
$F(95\%) > F$	insignificant

EXAMPLE: Comparison of several means

Sheet metal with a defined hardness from three suppliers is considered. With the test results of three samples, it should be explained whether a significant difference exists between the hardness values of the sheet metals of the suppliers.

	Outcomes y_{ij} of the Hardness Measurements	$\bar{y}_i$	s_i	s_i^2
Supplier 1	16 21 22 26 28 31 17 24 11 20 34 23 20 12 26	22.1	6.49	42.07
Supplier 2	26 29 24 18 27 27 21 36 28 20 32 32 22 27 24	26.2	4.90	24.03
Supplier 3	24 25 25 23 26 23 27 20 21 25 22 25 24 23 26	23.9	1.94	3.78

Mean variance: $\overline{s^2} = \dfrac{42.07 + 24.03 + 3.78}{3} = 23.29$

Variance of the three means: $s_{\bar{y}}^2 = 4.28$ *(pocket calculator!)*

Test statistic: $F = \dfrac{15 \cdot 4.28}{23.29} = 2.76$

Degrees of freedom: $f_1 = 3 - 1 = 2$, $f_2 = (15 - 1) \cdot 3 = 42$

Table values: $F(95\%) = 3.2$ $F(99\%) = 5.1$ $F < F(95\%)$ *(Table 8)*

No significant difference exists between the means of the sheet metal hardness.

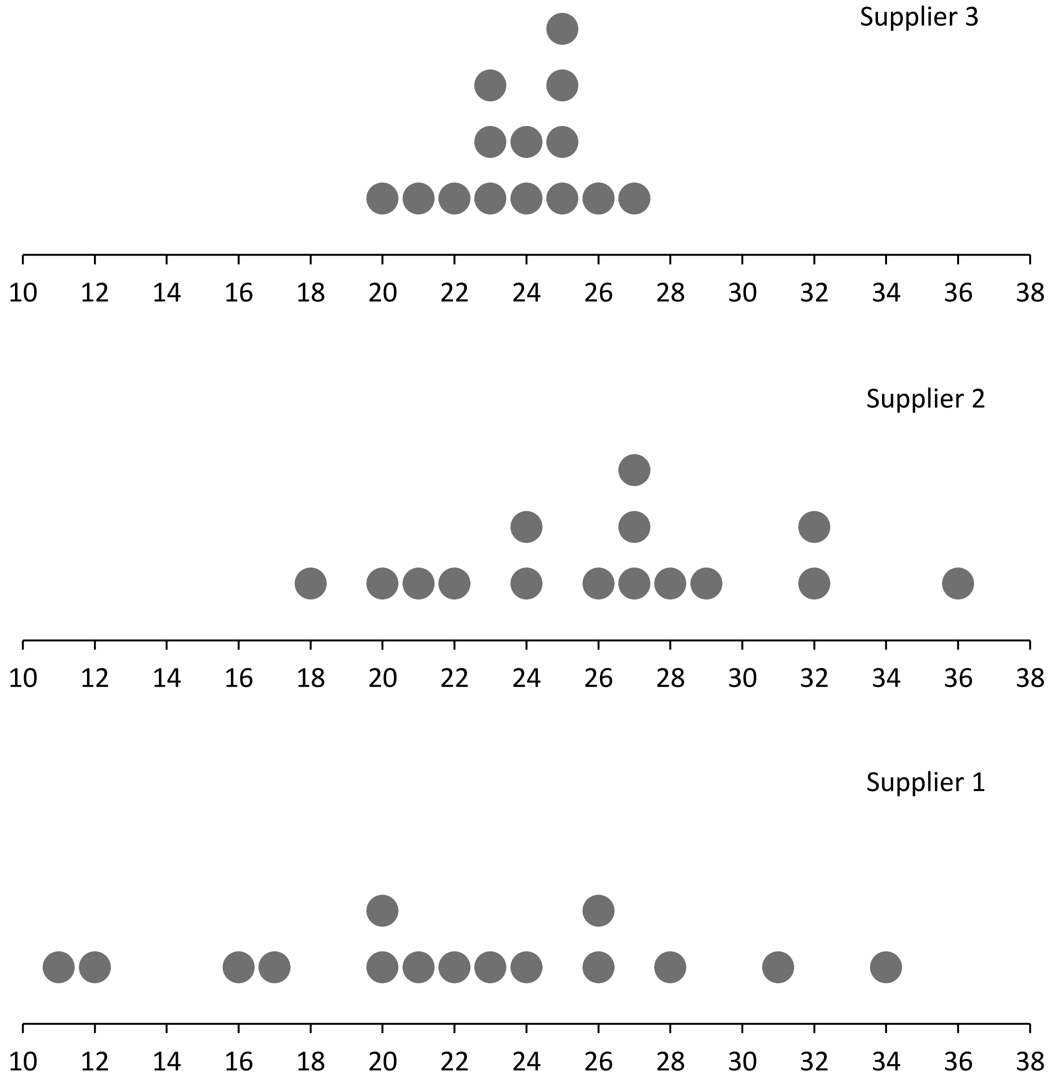

Fig. 14: Dot diagrams of the measurement results

There is clearly a difference in the variances of the three samples (compare with test for equality of several variances).

Comparison of Several Variances

Problem:

The outcome of an experimental investigation with k individual experiments, which were respectively repeated n times, is represented by k means $\bar{x}_i$ and k variances s_i^2:

Experiment No.	Outcomes	Mean	Variance
1	$x_{11}, x_{12}, \ldots, x_{1n}$	$\bar{x}_1$	s_1^2
2	$x_{21}, x_{22}, \ldots, x_{2n}$	$\bar{x}_2$	s_2^2
.	.	.	.
.	.	.	.
.	.	.	.
k	$x_{k1}, x_{k2}, \ldots, x_{kn}$	$\bar{x}_k$	s_k^2

It is to be decided whether the variances s_i^2 of the k experiments are significantly different or whether the differences are only coincidental.

Input quantities: x_{ij}, $\bar{x}_i$, s_i^2

Procedure:

1. Calculate by rows the absolute deviation of the measurement outcomes x_{ij} from the means $\bar{x}_i$. This corresponds to a conversion according to the rule

   $y_{ij} = \left| x_{ij} - \bar{x}_i \right|$.

2. Record the converted values y_{ij} corresponding to the above table and calculate the means $\bar{y}_i$ and the variances s_i^2.

Experiment No.	Outcomes	Mean	Variance
1	$y_{11}, y_{12}, \ldots, y_{1n}$	$\bar{y}_1$	s_1^2
2	$y_{21}, y_{22}, \ldots, y_{2n}$	$\bar{y}_2$	s_2^2
.	.	.	.
.	.	.	.
.	.	.	.
k	$y_{k1}, y_{k2}, \ldots, y_{kn}$	$\bar{y}_k$	s_k^2

3. Conduct the procedure for the comparison of several means (p. 38) with the assistance of the means $\bar{y}_i$ and variances s_i^2 of the converted values, i.e. F test with the test characteristic $F = \frac{n \cdot s_{\bar{y}}^2}{\overline{s_y^2}}$. Degrees of freedom: $f_1 = k-1$, $f_2 = (n-1) \cdot k$ (Table 8)

Outcome:

If F is greater than the quantile $F_{k-1;(n-1)\cdot k;0.99}$, for example, with a significance level of 1 % it is concluded that the variances s_i^2 of the original values x_{ij} are different.

REMARK:

Within the scope of tests on robust designs (or processes), it is frequently of interest to find settings of test parameters with which the test outcomes show a variability (variance) as small as possible.

For this purpose it makes sense to first examine, with the assistance of the test described above, whether the variances of the outcomes are significantly different in the individual experiment rows (compare with factorial variance analysis).

EXAMPLE: Test for the equality of several variances

Assumption of the values $\bar{x}_1 = 22.07$, $\bar{x}_2 = 26.2$, $\bar{x}_3 = 23.93$ *and* x_{ij} *from p. 39*

Conversion of the measurement values: $y_{ij} = \left| x_{ij} - \bar{x}_i \right|$

	Outcomes of the Conversion $y_{ij} = \left\| x_{ij} - \bar{x}_i \right\|$							
Supplier 1	6.07	1.07	0.07	3.93	5.93	8.93	5.07	1.93
	11.07	2.07	11.93	0.93	2.07	10.07	3.93	
Supplier 2	0.2	2.8	2.2	8.2	0.8	0.8	5.2	9.8
	1.8	6.2	5.8	5.8	4.2	0.8	2.2	
Supplier 3	0.07	1.07	1.07	0.93	2.07	0.93	3.07	3.93
	2.93	1.07	1.93	1.07	0.07	0.93	2.07	

Evaluation	$\bar{y}_i$	s_y	s_y^2
Supplier 1	5.0	3.9	15.23
Supplier 2	3.79	2.94	8.67
Supplier 3	1.55	1.1	1.22

Mean of the variances: $\overline{s_y^2} = \frac{s_1^2 + s_2^2 + s_3^2}{3} = \frac{15.23 + 8.67 + 1.22}{3} = 8.37$

Variance of the means: $s_{\bar{y}}^2 = 3.06$ *(pocket calculator!)*

Test statistic: $F = \frac{n \cdot s_{\bar{y}}^2}{\overline{s_y^2}} = \frac{15 \cdot 3.06}{8.37} = 5.5$ *Degrees of freedom:* $f_1 = k - 1 = 2$

$f_2 = (n-1) \cdot k = 14 \cdot 3 = 42$

Table value: $F_{2;42;0.99} \approx 5.1$ *(Table 8)* $F > F_{Table}$

Test decision:

The sheet metal hardness of the three suppliers have highly significant different variances.

Factorial Analysis of Variance

Problem:

On the basis of an experimental design with k rows and n experiments per row, factors were examined at two respective levels. The outcomes can be represented, in general, as they appear in the following example of an assessment matrix of a four-row plan for the columns of the factors A and B as well as the columns of the interaction AB.

Experiment No.	Factors A	Factors B	Factors AB	Outcomes y_{ij}	Means $\bar{y}_i$	Variances s_i^2
1	-	-	+	$y_{11}, y_{12}, \ldots, y_{1n}$	$\bar{y}_1$	s_1^2
2	+	-	-	$y_{21}, y_{22}, \ldots, y_{2n}$	$\bar{y}_2$	s_2^2
3	-	+	-	$y_{31}, y_{32}, \ldots, y_{3n}$	$\bar{y}_3$	s_3^2
4	+	+	+	$y_{41}, y_{42}, \ldots, y_{4n}$	$\bar{y}_4$	s_4^2

It is to be decided for each factor X whether it has a significant influence on the mean $\bar{y}_i$ of the experiment outcome.

Input quantities:

Means $\bar{y}_i$ and variances s_i^2 of the outcome of each row, number of rows k, number of experiments per row n

Procedure:

1. Calculate the average variance $\overline{s_y^2}$

   $$\overline{s_y^2} = \frac{1}{k} \cdot \sum_{i=1}^{k} s_i^2 .$$

   This quantity is a measure for the “experimental noise”.

2. Calculate the variance $s_{\bar{x}}^2$ of the means of the measurement values per level of factor X (see remark on the next page). The mean of all measurement outcomes is calculated, then, in which factor X (in the corresponding column) has the level “+” and the mean of all measurement outcomes in which the factor X has the level “-”.

   The variance $s_{\bar{x}}^2$ is calculated from these two means.

3. Calculate the test statistic

   $$F = \frac{s_{\bar{x}}^2 \cdot \text{number of measurement values per level}}{\overline{s_y^2}} .$$

4. Compare the test statistic F with the table values $F(95\%)$ and $F(99\%)$ to the degrees of freedom $f_1 = \text{number of levels} - 1$ and $f_2 = (n-1) \cdot k$ (Table 8).

   In the case of a two-level design (see above), $f_1 = 1$.

Test outcome:

If ...	... the statistical influence of factor X on the measurement outcome (means) is
$F \geq F(99\%)$	highly significant
$F(99\%) > F \geq F(95\%)$	significant
$F(95\%) > F$	insignificant

REMARK:

Each factor X corresponds to a column in the evaluation matrix. The procedure described is to be conducted separately for each individual column.

A detailed explanation of this assessment method is a component of the seminar on "Design of experiments (DOE)".

Depending on the size of the design present and the number of experiments per row, the manual assessment can be quite laborious. Because the risk of calculation error exists as well, the use of a suitable computer program is recommended.

It is recommended to first conduct a simple analysis of variance (ANOVA) before the factorial ANOVA in order to determine whether the outcomes in the experiment rows are at all significantly different. If this is not the case, the conducting of a factorial ANOVA is, naturally, not sensible.

EXAMPLE: Factorial analysis of variance

The electrode gap and the drawing dimension "C" of a spark plug is to be investigated with regard to its influence on the ignition voltage.

For this, a 2^2 design was conducted under defined laboratory conditions in which the electrode gap (factor A) and the drawing dimension "C" (factor B) were varied at two respective levels:

	Level "-"	Level "+"
Factor A	0.9 mm	1.0 mm
Factor B	0.2 mm	0.4 mm

Experiment No.	Factors A	B	AB	Measurement outcomes y_{ij} Ignition voltage in kV					
1	-	-	+	10.59	11.00	9.91	11.25	11.43	9.9
2	+	-	-	13.41	12.31	12.43	10.68	12.04	11.51
3	-	+	-	16.32	17.70	15.16	14.62	14.26	14.85
4	+	+	+	17.25	18.08	17.52	16.2	17.29	16.93

Evaluation					
Experi-ment No.	*Factors* A	B	AB	*Means* $\bar{y}_i$	*Variances* s_i^2
1	-	-	+	10.680	0.4398
2	+	-	-	12.063	0.8458
3	-	+	-	15.485	1.6722
4	+	+	+	17.212	0.3919

First, a simple analysis of variance is conducted:

Average variance: $\overline{s_y^2} = \frac{1}{4} \cdot (0.4398 + 0.8458 + 1.6722 + 0.3919) = 0.8374$

Total mean: $\bar{\bar{y}} = \frac{1}{4} \cdot (10.68 + 12.063 + 15.485 + 17.212) = 13.86$

Variance of the means: $s_{\bar{y}}^2 = \frac{1}{4-1} \cdot \sum_{i=1}^{4} (\bar{y}_i - \bar{\bar{y}})^2 = 9.0716$ *(pocket calculator!)*

Test statistic: $F = \frac{6 \cdot s_{\bar{y}}^2}{\overline{s_y^2}} = \frac{6 \cdot 9.0716}{0.8374} = 65$ $\quad f_1 = 4-1 = 3$ *and* $f_2 = (6-1) \cdot 4 = 20$

Table value: $F(99\%) = 4.94$ *(Table 8)* $F > F(99\%)$

The difference of the means in the four experiments is highly significant.

Assessment of factor A:

Mean of all means in which A is on the level "+": $\frac{12.063 + 17.212}{2} = 14.6375$

Mean of all means in which A is on the level "-": $\frac{10.68 + 15.485}{2} = 13.0825$

The variance of these two means is $s_{\bar{x}}^2(A) = \frac{(14.6375 - 13.0825)}{2} = 1.209$.

The number of measured values per level of factor A is 12. The degrees of freedom for the F test are $f_1 = 2-1 = 1$ *and* $f_2 = (6-1) \cdot 4 = 20$ *(this also applies to B and AB).*

$F = \frac{1.209 \cdot 12}{0.8374} = 17.32$ *Table value:* $F(99\%) = 8.1$ $F > F(99\%)$

The influence of the electrode gap on the ignition voltage is, then, highly significant.

Assessment of factor B:

Mean of all means in which B is on the level "+": $\frac{15485+17.212}{2}=16.3485$

Mean of all means in which B is on the level "-": $\frac{10.68+12.063}{2}=11.3715$

The variance of both of these means is $s_{\bar{x}}^2(B)=\frac{(16.3485-11.3715)^2}{2}=12.385$.

$F=\frac{12.385\cdot 12}{0.8374}=177.5$ *Table value:* $F(99\%)=8.1$ $F>F(99\%)$

The influence of the drawing dimension "C" to the ignition voltage is, then, highly significant.

Assessment of the interaction AB:

Mean of all means in which AB is on the level "+": $\frac{10.68+17.212}{2}=13.946$

Mean of all means in which AB is on the level "-": $\frac{12.063+15.485}{2}=13.774$

The variance of both of these means is $s_{\bar{x}}^2(AB)=\frac{(13.774-13.946)^2}{2}=0.0148$.

$F=\frac{0.0148\cdot 12}{0.8374}=0.21$ $f_1=2-1=1$ *and* $f_2=(6-1)\cdot 4=20$

Table value: $F(95\%)=4.35$ $F<F(95\%)$

It is obvious that no significant interaction exists between the electrode gap and the drawing dimension "C".

Outcome:

A value as high as possible was striven for for the target quantity of ignition voltage in this experiment. If the means of all means $\bar{y}_i$ *are respectively taken into consideration in which a factor was set on the lower level ("-") or upper level ("+"), it becomes immediately clear which factor setting is the more favorable with respect to the target quantity optimization. In this example, the upper level ("+") is to be selected, then, with both factors A and B.*

Regression Line

Problem:

n ordered pairs (x_i; y_i) are given. A recording of the points belonging to these ordered pairs in an x-y diagram (correlation diagram) indicates that it makes sense to draw an approximating line (regression line) through this cluster of points.

Slope b and axis section (intercept) a of the compensating line $y = a + b \cdot x$ are to be calculated.

Input quantities:

x_i, y_i, $\bar{x}$, $\bar{y}$, s_x^2, n

Procedure:

1. Calculation of slope b:

$$b = \frac{\frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y})}{s_x^2} = \frac{s_{xy}}{s_x^2}$$

The expression s_{xy} is called the covariance and can be calculated with the formula

$$s_{xy} = \frac{1}{n-1} \cdot \left(\sum_{i=1}^{n} x_i \cdot y_i - n \cdot \bar{x} \cdot \bar{y} \right)$$

(compare with “correlation” on page 50).

2. Calculation of a:

$$a = \bar{y} - b \cdot \bar{x}$$

REMARK:

a and b are determined in such a manner that the sum of the vertical quadratic deviations of measurement values y_i from the values $y(x_i)$ given by the regression line are minimal; the procedure is therefore called the “least-squares method”.

There are situations in which it is unclear whether the first value or the second value of a value couple will be respectively allocated to the x-axis. Here, both a recording of the points (x_i; y_i) as well as a recording of the points (y_i; x_i) is possible. For both of these cases, in general, two different regression lines are produced.

EXAMPLE: Calculation of a regression line

The torque M of a starter is measured with dependence on current I. A regression line $y = a + b \cdot x$ is to be calculated for 30 points (x_i; y_i).

Current I / A (Values x_i)									
110	122	125	132	136	148	152	157	167	173
182	187	198	209	219	220	233	245	264	273
281	295	311	321	339	350	360	375	390	413
Torque M / Nm (Values y_i)									
1,80	1,46	1,90	1,76	2,45	2,28	2,67	2,63	2,85	3,15
3,10	3,52	3,85	3,85	4,07	4,69	4,48	4,96	5,55	5,34
5,95	6,45	6,52	6,84	7,55	7,27	7,80	7,90	8,65	8,65

Allocation of the values:

(x_1 ; y_1) = (110; 1.80)   (x_2 ; y_2) = (122; 1.46)   (x_{11}; y_{11}) = (182; 3.10) etc.

The following is produced: $\bar{x} = 236.233$ $\bar{y} = 4.665$ $s_x^2 = 8060.6$ $n = 30$

$$s_{xy} = \frac{110 \cdot 1.8 + 122 \cdot 1.46 + \ldots + 413 \cdot 8.65 - 30 \cdot 236.2 \cdot 4.67}{29} = 200.372 .$$

One then finds: $b = \frac{200.372}{8060.6} = 0.025$

and $a = 4.665 - 0.025 \cdot 236.233 = -1.208$.

Equation of the straight line $y = -1.208 + 0.025 \cdot x$

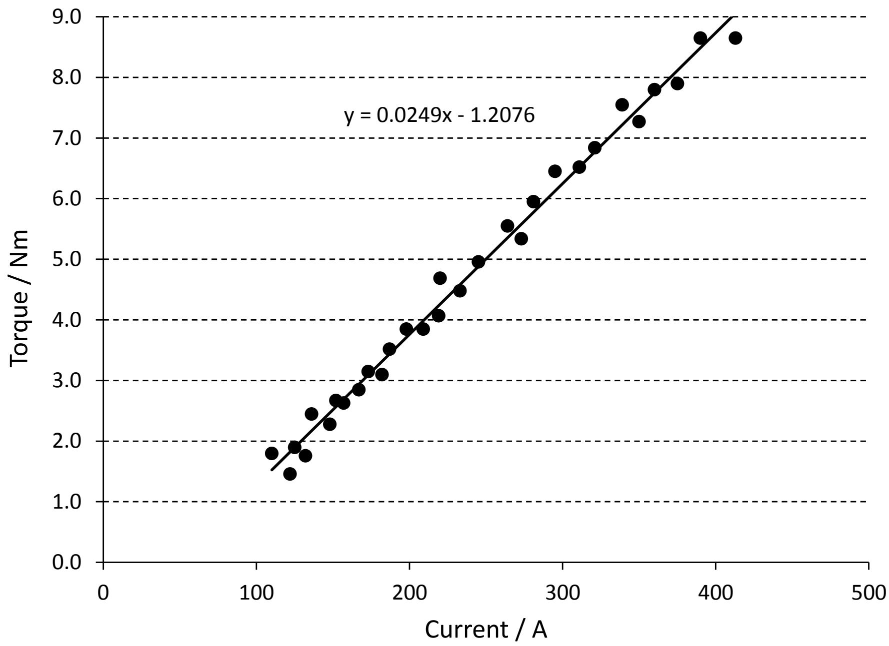

Fig. 15: x-y diagram

Correlation Coefficient

Problem:

n ordered pairs $(x_i;\ y_i)$ of two quantities X and Y are given. A recording of the points belonging to these ordered pairs in a correlation diagram (x-y diagram) shows a cluster of points which allows the presumption that a functional association (a correlation) exists between quantities X and Y.

The correlation coefficient r, a measure for the “quality” of the presumed association, is to be calculated.

Input quantities:

$x_i,\ y_i,\ \bar{x},\ \bar{y},\ s_x,\ s_y,\ n$

Procedure:

Calculation of r:

$$r = \frac{\frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y})}{s_x \cdot s_y} = \frac{s_{xy}}{s_x \cdot s_y}$$

Remark:

r can assume values between -1 and +1. A value in the proximity of +1 (-1), e.g. 0,9 (-0,9), corresponds to a cluster of points which can be approximated quite well with a regression line with positive (negative) slope; one then refers to a strong positive (negative) correlation.

A strong correlation does not inevitably signify that Y is directly dependent on X; it can be produced by the dependence of both quantities X and Y on a third quantity Z (illusory correlation).

The expression s_{xy} in the numerator is called the covariance of the sample. It can also be calculated with the assistance of the formula

$$s_{xy} = \frac{1}{n-1} \cdot \left(\sum_{i=1}^{n} x_i \cdot y_i - n \cdot \bar{x} \cdot \bar{y} \right).$$

In the example represented on the right, it is clear in advance that there is a correlation between X and Y. Here, the central question is that for the degree of the correlation. It is clear that the means of both data sets are different (one should observe the scaling!).

EXAMPLE: Calculation of the correlation coefficient

The flow rate of 25 injectors was measured at two different test benches with different test media (test fluids):

Test bench 1, Fluid 1, measured values x_i									
Flow rate (mass) per 1000 strokes in g									
4.50	4.54	4.56	4.61	4.56	4.42	4.60	4.46	4.56	4.50
4.50	4.54	4.53	4.55	4.49	4.49	4.45	4.52	4.57	4.59
4.52	4.60	4.50	4.54	4.61					

Test bench 2, Fluid 2, measured values y_i									
Flow rate (mass) per 1000 strokes in g									
4.87	4.92	4.94	4.95	4.95	4.81	4.95	4.83	4.90	4.91
4.92	4.90	4.85	4.90	4.88	4.85	4.84	4.87	4.92	4.94
4.89	4.96	4.87	4.92	4.94					

$\bar{x} = 4.5324 \qquad \bar{y} = 4.8992 \qquad s_x = 0.0506 \qquad s_y = 0.042 \qquad n = 25$

$$\sum_{i=1}^{n} x_i \cdot y_i = 4.50 \cdot 4.87 + 4.54 \cdot 4.92 + \ldots + 4.61 \cdot 4.94 = 555.1733$$

$$s_{xy} = \frac{555.1733 - 25 \cdot 4.5324 \cdot 4.8992}{24} = 0.00187$$

$$r = \frac{0.00187}{0.0506 \cdot 0.042} = 0.88$$

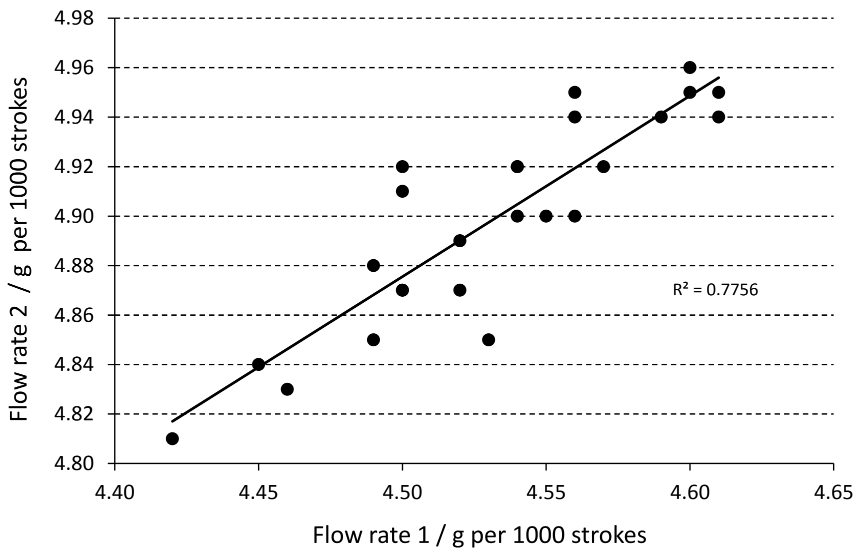

Fig. 16: x-y diagram

Law of Error Propagation

The Gaussian law of error propagation describes how the errors of measurement of several independent measurands x_i influence a target quantity z, which is calculated according to a functional relation $z = f(x_1, x_2, \dots, x_k)$:

$$s^2 \approx \left(\frac{\partial z}{\partial x_1}\right)^2 \cdot s_1^2 + \left(\frac{\partial z}{\partial x_2}\right)^2 \cdot s_2^2 + \dots + \left(\frac{\partial z}{\partial x_k}\right)^2 \cdot s_k^2 .$$

z is then, in general, only an indirectly measurable quantity. For example, the area of a rectangle can be determined in which the side lengths are measured and the measurement outcomes are multiplied with each other.

The accuracy with which z can be given is dependent on the accuracy of the measurands x_i. In general, the mean $\bar{x}_i$ and the standard deviation s_i of a sequence of repeated measurements are respectively given for x_i.

In the above expression, $\frac{\partial z}{\partial x_i}$ designates the partial derivations of the function z according to the variables x_i. They are to be respectively calculated at the positions $\bar{x}_i$. The derivation of the formula for s_z contains the development of z in a Taylor's series and the neglecting of terms of a higher order. For this reason, the approximation symbol takes the place of the equal symbol, designating the approximate equality.

The use of the law of error propagation is most easily comprehensible on the basis of an example.

We are considering two resistors at which a sequence of repeated measurements are respectively conducted. As the outcome of these measurement series, the mean and standard deviation are respectively given as the measure for the average error:

$R_1 = 47 \pm 0.8\,\Omega$ $\qquad$ $R_2 = 68 \pm 1.1\,\Omega$.

In the case of the parallel connection, the total resistance R is to be calculated according to

$$R = \frac{R_1 \cdot R_2}{R_1 + R_2} : \qquad R = \frac{47 \cdot 68\,\Omega}{47 + 68} = 27.8\,\Omega .$$

In the calculation of the corresponding average error, the partial derivations of R according to R_1 and R_2 are first required:

$$\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1 + R_2)^2} = \frac{68^2}{(47+68)^2} = 0.35$$

$$\frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1 + R_2)^2} = \frac{47^2}{(47+68)^2} = 0.167 .$$

The average error of the total resistance is calculated using the law of error propagation:

$$s_R^2 = 0.35 \cdot (0.8\,\Omega)^2 + 0.167^2 \cdot (1.1\,\Omega)^2 = 0.11\,\Omega^2 \qquad \Rightarrow s_R = 0.33\,\Omega .$$

The corresponding result is: $R = 27.8 \pm 0.3\,\Omega$.

The law of error propagation can be represented very easily if the function f, which designates the association of the independent measurands x_i, is a sum.

The partial derivations are, then, all equal to 1.0 and the following is produced:

$$s_z^2 = s_1^2 + s_2^2 + \ldots + s_k^2 .$$

Correspondingly, the following applies for the variances of the corresponding total population:

$$\sigma_z^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_k^2 .$$

This means that the variance of an indirect measurand, which is determined by addition of independent individual measurands, is equal to the sum of the variances of the individual measurands.

HINT:

The above equation is identical to a general relationship between the variances of several independent random variables. It can also be derived completely independent of the law of error propagation solely on the basis of statistical calculation rules.

EXAMPLE:

In the case of a series connection of the two resistors, one finds:

$$s_R^2 = s_{R_1}^2 + s_{R_2}^2 = (0.8\,\Omega)^2 + (1.1\,\Omega)^2 \Leftrightarrow s_R = 1.36\,\Omega$$

that is $R = 115 \pm 1.4\,\Omega$.

The Weibull Probability Plot

Problem:

The failure behavior of products can generally be described with the Weibull distribution. Important parameters of this distribution are the characteristic service life T and the shape parameter b. Estimated values of both of these parameters should be determined by recording of the measured failure times t_i of n products (components) in a Weibull plot.

Input quantities:

Times t_i up to the failure of the i-th product

Procedure:

1.1 Determination of the cumulative frequencies H_j with a large sample $n > 50$.

- Subdivide the total observation time into k classes (compare to histogram).
- Determine the absolute frequencies n_j of the failure times falling into the individual classes by counting out.
- Calculate the cumulative frequency $G_j = \sum_{i=1}^{j} n_i$ for $j = 1, 2, \ldots, k$

  i.e. $G_1 = n_1$, $G_2 = n_1 + n_2$, . . . , $G_j = n_1 + n_2 + \ldots + n_j$.
- Calculate the relative cumulative frequency $H_j = \frac{G_j}{n} \cdot 100\%$ for each class.

1.2 Determination of the cumulative frequencies H_j with a small sample $6 < n \leq 50$.

- Arrange the n individual values in order of magnitude.

  The smallest value has the rank of 1, the largest has the rank of n.
- Allocate the tabulated relative cumulative frequencies H_j in Table 1 for sample size n to each rank number j.

2. Record the determined relative cumulative frequencies H_j over the logarithmic time axis (t-axis). In case 1.1, the recording occurs via the upper class limits (the highest class is inapplicable here), and in case 1.2 it occurs via the individual values t_i.
3. Draw an approximating line through the points. If the points drawn-in lie adequately well on the straight line, one can conclude from this that the present failure characteristic can be described quite well by the Weibull distribution. Otherwise, a determination of T and b is not reasonable.
4. Go to the point of intersection of the straight line and the horizontal 63.2 % line. Drop a perpendicular on the time axis. The characteristic life T can there be read.
5. Draw a line parallel to the approximating line through the pole (here 1E4; 63.2 %) of the b scale. The shape parameter b (slope) can then be read there.

REMARK:

The scaling of the logarithmic time axis (t-axis) can be adjusted through the multiplication with a suitable scale factor of the present failure times. If one selects, for example, the factor 10h (10 hours), the following corresponds:

the number 1 on the t-axis to the value 10 hours,

the number 10 on the t-axis to the value 100 hours,

the number 100 on the t-axis to the value 1000 hours.

The scaling can also, for example, be conducted for the mileage, the number of load cycles, switching sequences or work cycles, instead of being conducted for the time.

EXAMPLE:

A sample of 10 electric motors was examined according to a fixed test cycle at a continuous test bench. Until the respective failure, the test objects achieved the following cycle counts (in increasing sequence and in units of 10^5 cycles):

Motor No.	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Cycles until failure / 10^5	*0.41*	*0.55*	*0.79*	*0.92*	*1.1*	*1.1*	*1.4*	*1.5*	*1.8*	*1.9*
Rel. cumulative freq. in %	*6.2*	*15.9*	*25.5*	*35.2*	*45.2*	*54.8*	*64.8*	*74.5*	*84.1*	*93.8*

Characteristic life $T = 1.31 \cdot 10^5$ test cycles, Shape factor $b = 2.27$

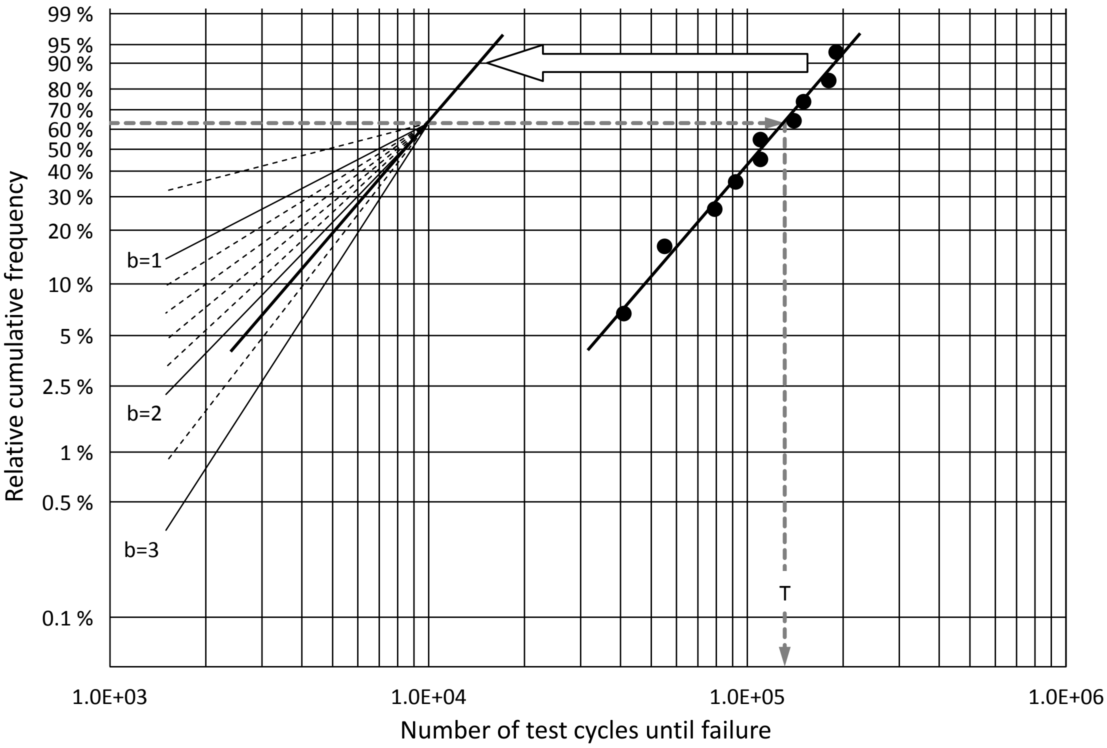

Fig. 17: Representation of the points in a Weibull plot: x-axis (= t-axis)

HINT 1:

Depending on the type of Weibull plot (as a paper form or within a software) the reference scale for b can be realized in various ways, e.g. in top left or bottom right corner or as a quarter circle.

HINT 2:

With the assistance of conversions $x_i = ln(t_i)$ *and* $y_i = ln(ln(1/(1-H_i)))$, b *and* T *can also be directly calculated:* $b = \frac{s_{xy}}{s_x^2}$ *and* $T = e^{\bar{x} - \frac{\bar{y}}{b}}$.

Here, $\bar{x}$ *and* s_x *are the mean or the standard deviation of* x_i, $\bar{y}$ *is the mean of* y_i *and* s_{xy} *is the covariance (compare with the section on regression lines) of* x_i *and* y_i *(with* $i = 1 \ldots k$ *in case 1.1 and* $i = 1 \ldots n$ *in case 1.2).*

Tables

Table 1

Cumulative frequencies $H_i(n)$ (in percent) for a probability plot of the points (x_i, H_i) from ranked samples

i	n=6	n=7	n=8	n=9	n=10	n=11	n=12	n=13	n=14	n=15
1	10.2	8.9	7.8	6.8	6.2	5.6	5.2	4.8	4.5	4.1
2	26.1	22.4	19.8	17.6	15.9	14.5	13.1	12.3	11.3	10.6
3	42.1	36.3	31.9	28.4	25.5	23.3	21.5	19.8	18.4	17.1
4	57.9	50.0	44.0	39.4	35.2	32.3	29.5	27.4	25.5	23.9
5	73.9	63.7	56.0	50.0	45.2	41.3	37.8	34.8	32.3	30.2
6	89.8	77.6	68.1	60.6	54.8	50.0	46.0	42.5	39.4	36.7
7		91.2	80.2	71.6	64.8	58.7	54.0	50.0	46.4	43.3
8			92.2	82.4	74.5	67.7	62.2	57.5	53.6	50.0
9				93.2	84.1	76.7	70.5	65.2	60.6	56.7
10					93.8	85.5	78.5	72.6	67.7	63.3
11						94.4	86.9	80.2	74.5	69.8
12							94.9	87.7	81.6	76.1
13								95.3	88.7	82.9
14									95.5	89.4
15										95.9

The cumulative frequency $H_i(n)$ for rank number i can also be calculated with the approximation formulas

$$H_i(n) = \frac{i - 0.5}{n} \qquad \text{and} \qquad H_i(n) = \frac{i - 0.3}{n + 0.4}.$$

The deviation from exact table values is hereby insignificant.

EXAMPLE:

$n = 15$ $\quad i = 12$ $\quad$ Table value: 76.1

$$H_{12}(15) = \frac{12 - 0.5}{15} = 76.7 \qquad \text{or} \qquad H_{12}(15) = \frac{12 - 0.3}{15 + 0.4} = 76.0$$

Table 1 (cont.)

i	n=16	n=17	n=18	n=19	n=20	n=21	n=22	n=23	n=24	n=25
1	3.9	3.7	3.4	3.3	3.1	2.9	2.8	2.7	2.6	2.4
2	10.0	9.3	8.9	8.4	7.9	7.6	7.2	6.9	6.7	6.4
3	16.1	15.2	14.2	13.6	12.9	12.3	11.7	11.3	10.7	10.4
4	22.4	20.9	19.8	18.7	17.9	17.1	16.4	15.6	14.9	14.2
5	28.4	26.8	25.1	23.9	22.7	21.8	20.6	19.8	18.9	18.1
6	34.8	32.6	30.9	29.1	27.8	26.4	25.1	24.2	23.3	22.4
7	40.9	38.2	36.3	34.5	32.6	31.2	29.8	28.4	27.4	26.1
8	46.8	44.0	41.7	39.7	37.8	35.9	34.1	32.6	31.6	30.2
9	53.2	50.0	47.2	44.8	42.5	40.5	38.6	37.1	35.6	34.1
10	59.1	56.0	52.8	50.0	47.6	45.2	43.3	41.3	39.7	38.2
11	65.2	61.8	58.3	55.2	52.4	50.0	47.6	45.6	43.6	42.1
12	71.6	67.4	63.7	60.3	57.5	54.8	52.4	50.0	48.0	46.0
13	77.6	73.2	69.1	65.5	62.2	59.5	56.7	54.4	52.0	50.0
14	83.9	79.1	74.9	70.9	67.4	64.1	61.4	58.7	56.4	54.0
15	90.0	84.8	80.2	76.1	72.2	68.8	65.9	62.9	60.3	57.9
16	96.1	90.7	85.8	81.3	77.3	73.6	70.2	67.4	64.4	61.8
17		96.3	91.2	86.4	82.1	78.2	74.9	71.6	68.4	65.9
18			96.6	91.6	87.1	82.9	79.4	75.8	72.6	69.8
19				96.7	92.1	87.7	83.6	80.2	76.7	73.9
20					96.9	92.4	88.3	84.4	81.1	77.6
21						97.1	92.8	88.7	85.1	81.9
22							97.2	93.1	89.3	85.8
23								97.3	93.3	89.6
24									97.4	93.6
25										97.6

Table 1 (cont.)

i	n=26	n=27	n=28	n=29	n=30	n=31	n=32	n=33	n=34	n=35
1	2.4	2.3	2.2	2.1	2.1	2.0	1.9	1.9	1.8	1.8
2	6.2	5.9	5.7	5.5	5.3	5.1	5.0	4.8	4.7	4.6
3	9.9	9.5	9.2	8.9	8.7	8.3	8.1	7.8	7.6	7.4
4	13.8	13.4	12.7	12.3	11.9	11.6	11.2	10.8	10.5	10.2
5	17.6	16.9	16.4	15.9	15.2	14.8	14.3	13.9	13.5	13.1
6	21.5	20.6	19.8	19.2	18.7	18.0	17.4	16.9	16.4	15.9
7	25.1	24.2	23.3	22.7	21.8	21.2	20.5	19.9	19.3	18.8
8	29.1	28.1	27.1	26.1	25.1	24.4	23.6	22.9	22.2	21.6
9	33.0	31.6	30.5	29.5	28.4	27.6	26.7	25.9	25.1	24.5
10	36.7	35.2	34.1	33.0	31.9	30.8	29.8	28.9	28.1	27.3
11	40.5	39.0	37.4	36.3	35.2	34.0	32.9	31.9	31.0	30.1
12	44.4	42.5	41.3	39.7	38.6	37.2	36.0	34.9	33.9	33.0
13	48.0	46.4	44.8	43.3	41.7	40.4	39.1	37.9	36.8	35.8
14	52.0	50.0	48.4	46.4	45.2	43.6	42.2	41.0	39.8	38.6
15	55.6	53.6	51.6	50.0	48.4	46.8	45.3	44.0	42.7	41.5
16	59.5	57.5	55.2	53.6	51.6	50.0	48.4	47.0	45.6	44.3
17	63.3	61.0	58.7	56.7	54.8	53.2	51.6	50.0	48.5	47.2
18	67.0	64.8	62.6	60.3	58.3	56.4	54.7	53.0	51.5	50.0
19	70.9	68.4	65.9	63.7	61.4	59.6	57.8	56.0	54.4	52.8
20	74.9	71.9	69.5	67.0	64.8	62.8	60.9	59.0	57.3	55.7
21	78.5	75.8	72.9	70.5	68.1	66.0	64.0	62.1	60.2	58.5
22	82.4	79.4	76.7	73.9	71.6	69.2	67.1	65.1	63.2	61.4
23	86.2	83.1	80.2	77.3	74.9	72.4	70.2	68.1	66.1	64.2
24	90.2	86.6	83.6	80.8	78.2	75.6	73.3	71.1	69.0	67.0
25	93.8	90.5	87.3	84.1	81.3	78.8	76.4	74.1	71.9	69.9
26	97.6	94.1	90.8	87.7	84.8	82.0	79.5	77.1	74.9	72.7
27		97.7	94.3	91.2	88.1	85.3	82.6	80.1	77.8	75.6
28			97.8	94.5	91.3	88.5	85.7	83.1	80.7	78.4
29				97.9	94.7	91.7	88.8	86.2	83.6	81.3
30					97.9	94.9	91.9	89.2	86.5	84.1
31						98.0	95.0	92.2	89.5	86.9
32							98.1	95.2	92.4	89.8
33								98.1	95.3	92.6
34									98.2	95.5
35										98.2

Table 2

On the test of normality (according to Pearson)

	Level of significance $\alpha = 0{,}5\,\%$		Level of significance $\alpha = 2{,}5\,\%$	
Sample size n	Lower limit	Upper limit	Lower limit	Upper limit
3	1.735	2.000	1.745	2.000
4	1.83	2.447	1.93	2.439
5	1.98	2.813	2.09	2.782
6	2.11	3.115	2.22	3.056
7	2.22	3.369	2.33	3.282
8	2.31	3.585	2.43	3.471
9	2.39	3.772	2.51	3.634
10	2.46	3.935	2.59	3.777
11	2.53	4.079	2.66	3.903
12	2.59	4.208	2.72	4.02
13	2.64	4.325	2.78	4.12
14	2.70	4.431	2.83	4.21
15	2.74	4.530	2.88	4.29
16	2.79	4.62	2.93	4.37
17	2.83	4.70	2.97	4.44
18	2.87	4.78	3.01	4.51
19	2.90	4.85	3.05	4.57
20	2.94	4.91	3.09	4.63
25	3.09	5.19	3.24	4.87
30	3.21	5.40	3.37	5.06
35	3.32	5.57	3.48	5.21
40	3.41	5.71	3.57	5.34
45	3.49	5.83	3.66	5.45
50	3.56	5.93	3.73	5.54
55	3.62	6.02	3.80	5.63
60	3.68	6.10	3.86	5.70
65	3.74	6.17	3.91	5.77
70	3.79	6.24	3.96	5.83
75	3.83	6.30	4.01	5.88
80	3.88	6.35	4.05	5.93
85	3.92	6.40	4.09	5.98
90	3.96	6.45	4.13	6.03
95	3.99	6.49	4.17	6.07
100	4.03	6.53	4.21	6.11
150	4.32	6.82	4.48	6.39
200	4.53	7.01	4.68	6.60
500	5.06	7.60	5.25	7.15
1000	5.50	7.99	5.68	7.54

Table 3

Upper limit UL for the outlier test (David-Hartley-Pearson-Test)

	Level of significance α				
n	10 %	5 %	2.5 %	1 %	0.5 %
3	1.997	1.999	2.000	2.000	2.000
4	2.409	2.429	2.439	2.445	2.447
5	2.712	2.753	2.782	2.803	2.813
6	2.949	3.012	3.056	3.095	3.115
7	3.143	3.222	3.282	3.338	3.369
8	3.308	3.399	3.471	3.543	3.585
9	3.449	3.552	3.634	3.720	3.772
10	3.570	3.685	3.777	3.875	3.935
11	3.68	3.80	3.903	4.012	4.079
12	3.78	3.91	4.01	4.134	4.208
13	3.87	4.00	4.11	4.244	4.325
14	3.95	4.09	4.21	4.34	4.431
15	4.02	4.17	4.29	4.43	4.53
16	4.09	4.24	4.37	4.51	4.62
17	4.15	4.31	4.44	4.59	4.69
18	4.21	4.38	4.51	4.66	4.77
19	4.27	4.43	4.57	4.73	4.84
20	4.32	4.49	4.63	4.79	4.91
30	4.70	4.89	5.06	5.25	5.39
40	4.96	5.15	5.34	5.54	5.69
50	5.15	5.35	5.54	5.77	5.91
60	5.29	5.50	5.70	5.93	6.09
80	5.51	5.73	5.93	6.18	6.35
100	5.68	5.90	6.11	6.36	6.54
150	5.96	6.18	6.39	6.64	6.84
200	6.15	6.38	6.59	6.85	7.03
500	6.72	6.94	7.15	7.42	7.60
1000	7.11	7.33	7.54	7.80	7.99

Table 4 Standard normal distribution $\Phi(-u) = 1 - \Phi(u)$ $D(u) = \Phi(u) - \Phi(-u)$

u	Φ (-u)	Φ (u)	D(u)
0.01	0.496011	0.503989	0.007979
0.02	0.492022	0.507978	0.015957
0.03	0.488033	0.511967	0.023933
0.04	0.484047	0.515953	0.031907
0.05	0.480061	0.519939	0.039878
0.06	0.476078	0.523922	0.047845
0.07	0.472097	0.527903	0.055806
0.08	0.468119	0.531881	0.063763
0.09	0.464144	0.535856	0.071713
0.10	0.460172	0.539828	0.079656
0.11	0.456205	0.543795	0.087591
0.12	0.452242	0.547758	0.095517
0.13	0.448283	0.551717	0.103434
0.14	0.444330	0.555670	0.111340
0.15	0.440382	0.559618	0.119235
0.16	0.436441	0.563559	0.127119
0.17	0.432505	0.567495	0.134990
0.18	0.428576	0.571424	0.142847
0.19	0.424655	0.575345	0.150691
0.20	0.420740	0.579260	0.158519
0.21	0.416834	0.583166	0.166332
0.22	0.412936	0.587064	0.174129
0.23	0.409046	0.590954	0.181908
0.24	0.405165	0.594835	0.189670
0.25	0.401294	0.598706	0.197413
0.26	0.397432	0.602568	0.205136
0.27	0.393580	0.606420	0.212840
0.28	0.389739	0.610261	0.220522
0.29	0.385908	0.614092	0.228184
0.30	0.382089	0.617911	0.235823
0.31	0.378281	0.621719	0.243439
0.32	0.374484	0.625516	0.251032
0.33	0.370700	0.629300	0.258600
0.34	0.366928	0.633072	0.266143
0.35	0.363169	0.636831	0.273661
0.36	0.359424	0.640576	0.281153
0.37	0.355691	0.644309	0.288617
0.38	0.351973	0.648027	0.296054
0.39	0.348268	0.651732	0.303463
0.40	0.344578	0.655422	0.310843
0.41	0.340903	0.659097	0.318194
0.42	0.337243	0.662757	0.325514
0.43	0.333598	0.666402	0.332804
0.44	0.329969	0.670031	0.340063
0.45	0.326355	0.673645	0.347290
0.46	0.322758	0.677242	0.354484
0.47	0.319178	0.680822	0.361645
0.48	0.315614	0.684386	0.368773
0.49	0.312067	0.687933	0.375866
0.50	0.308538	0.691462	0.382925

u	Φ (-u)	Φ (u)	D(u)
0.51	0.305026	0.694974	0.389949
0.52	0.301532	0.698468	0.396936
0.53	0.298056	0.701944	0.403888
0.54	0.294598	0.705402	0.410803
0.55	0.291160	0.708840	0.417681
0.56	0.287740	0.712260	0.424521
0.57	0.284339	0.715661	0.431322
0.58	0.280957	0.719043	0.438085
0.59	0.277595	0.722405	0.444809
0.60	0.274253	0.725747	0.451494
0.61	0.270931	0.729069	0.458138
0.62	0.267629	0.732371	0.464742
0.63	0.264347	0.735653	0.471306
0.64	0.261086	0.738914	0.477828
0.65	0.257846	0.742154	0.484308
0.66	0.254627	0.745373	0.490746
0.67	0.251429	0.748571	0.497142
0.68	0.248252	0.751748	0.503496
0.69	0.245097	0.754903	0.509806
0.70	0.241964	0.758036	0.516073
0.71	0.238852	0.761148	0.522296
0.72	0.235762	0.764238	0.528475
0.73	0.232695	0.767305	0.534610
0.74	0.229650	0.770350	0.540700
0.75	0.226627	0.773373	0.546745
0.76	0.223627	0.776373	0.552746
0.77	0.220650	0.779350	0.558700
0.78	0.217695	0.782305	0.564609
0.79	0.214764	0.785236	0.570472
0.80	0.211855	0.788145	0.576289
0.81	0.208970	0.791030	0.582060
0.82	0.206108	0.793892	0.587784
0.83	0.203269	0.796731	0.593461
0.84	0.200454	0.799546	0.599092
0.85	0.197662	0.802338	0.604675
0.86	0.194894	0.805106	0.610211
0.87	0.192150	0.807850	0.615700
0.88	0.189430	0.810570	0.621141
0.89	0.186733	0.813267	0.626534
0.90	0.184060	0.815940	0.631880
0.91	0.181411	0.818589	0.637178
0.92	0.178786	0.821214	0.642427
0.93	0.176186	0.823814	0.647629
0.94	0.173609	0.826391	0.652782
0.95	0.171056	0.828944	0.657888
0.96	0.168528	0.831472	0.662945
0.97	0.166023	0.833977	0.667954
0.98	0.163543	0.836457	0.672914
0.99	0.161087	0.838913	0.677826
1.00	0.158655	0.841345	0.682689

u	Φ (-u)	Φ (u)	D(u)
1.01	0.156248	0.843752	0.687505
1.02	0.153864	0.846136	0.692272
1.03	0.151505	0.848495	0.696990
1.04	0.149170	0.850830	0.701660
1.05	0.146859	0.853141	0.706282
1.06	0.144572	0.855428	0.710855
1.07	0.142310	0.857690	0.715381
1.08	0.140071	0.859929	0.719858
1.09	0.137857	0.862143	0.724287
1.10	0.135666	0.864334	0.728668
1.11	0.133500	0.866500	0.733001
1.12	0.131357	0.868643	0.737286
1.13	0.129238	0.870762	0.741524
1.14	0.127143	0.872857	0.745714
1.15	0.125072	0.874928	0.749856
1.16	0.123024	0.876976	0.753951
1.17	0.121001	0.878999	0.757999
1.18	0.119000	0.881000	0.762000
1.19	0.117023	0.882977	0.765953
1.20	0.115070	0.884930	0.769861
1.21	0.113140	0.886860	0.773721
1.22	0.111233	0.888767	0.777535
1.23	0.109349	0.890651	0.781303
1.24	0.107488	0.892512	0.785024
1.25	0.105650	0.894350	0.788700
1.26	0.103835	0.896165	0.792331
1.27	0.102042	0.897958	0.795915
1.28	0.100273	0.899727	0.799455
1.29	0.098525	0.901475	0.802949
1.30	0.096801	0.903199	0.806399
1.31	0.095098	0.904902	0.809804
1.32	0.093418	0.906582	0.813165
1.33	0.091759	0.908241	0.816482
1.34	0.090123	0.909877	0.819755
1.35	0.088508	0.911492	0.822984
1.36	0.086915	0.913085	0.826170
1.37	0.085344	0.914656	0.829313
1.38	0.083793	0.916207	0.832413
1.39	0.082264	0.917736	0.835471
1.40	0.080757	0.919243	0.838487
1.41	0.079270	0.920730	0.841460
1.42	0.077804	0.922196	0.844392
1.43	0.076359	0.923641	0.847283
1.44	0.074934	0.925066	0.850133
1.45	0.073529	0.926471	0.852941
1.46	0.072145	0.927855	0.855710
1.47	0.070781	0.929219	0.858438
1.48	0.069437	0.930563	0.861127
1.49	0.068112	0.931888	0.863776
1.50	0.066807	0.933193	0.866386

u	Φ (-u)	Φ (u)	D(u)
1.51	0.065522	0.934478	0.868957
1.52	0.064256	0.935744	0.871489
1.53	0.063008	0.936992	0.873983
1.54	0.061780	0.938220	0.876440
1.55	0.060571	0.939429	0.878858
1.56	0.059380	0.940620	0.881240
1.57	0.058208	0.941792	0.883585
1.58	0.057053	0.942947	0.885893
1.59	0.055917	0.944083	0.888165
1.60	0.054799	0.945201	0.890401
1.61	0.053699	0.946301	0.892602
1.62	0.052616	0.947384	0.894768
1.63	0.051551	0.948449	0.896899
1.64	0.050503	0.949497	0.898995
1.65	0.049471	0.950529	0.901057
1.66	0.048457	0.951543	0.903086
1.67	0.047460	0.952540	0.905081
1.68	0.046479	0.953521	0.907043
1.69	0.045514	0.954486	0.908972
1.70	0.044565	0.955435	0.910869
1.71	0.043633	0.956367	0.912734
1.72	0.042716	0.957284	0.914568
1.73	0.041815	0.958185	0.916370
1.74	0.040929	0.959071	0.918141
1.75	0.040059	0.959941	0.919882
1.76	0.039204	0.960796	0.921592
1.77	0.038364	0.961636	0.923273
1.78	0.037538	0.962462	0.924924
1.79	0.036727	0.963273	0.926546
1.80	0.035930	0.964070	0.928139
1.81	0.035148	0.964852	0.929704
1.82	0.034379	0.965621	0.931241
1.83	0.033625	0.966375	0.932750
1.84	0.032884	0.967116	0.934232
1.85	0.032157	0.967843	0.935687
1.86	0.031443	0.968557	0.937115
1.87	0.030742	0.969258	0.938516
1.88	0.030054	0.969946	0.939892
1.89	0.029379	0.970621	0.941242
1.90	0.028716	0.971284	0.942567
1.91	0.028067	0.971933	0.943867
1.92	0.027429	0.972571	0.945142
1.93	0.026803	0.973197	0.946393
1.94	0.026190	0.973810	0.947620
1.95	0.025588	0.974412	0.948824
1.96	0.024998	0.975002	0.950004
1.97	0.024419	0.975581	0.951162
1.98	0.023852	0.976148	0.952297
1.99	0.023295	0.976705	0.953409
2.00	0.022750	0.977250	0.954500

u	Φ (-u)	Φ (u)	D(u)
2.01	0.022216	0.977784	0.955569
2.02	0.021692	0.978308	0.956617
2.03	0.021178	0.978822	0.957644
2.04	0.020675	0.979325	0.958650
2.05	0.020182	0.979818	0.959636
2.06	0.019699	0.980301	0.960602
2.07	0.019226	0.980774	0.961548
2.08	0.018763	0.981237	0.962475
2.09	0.018309	0.981691	0.963382
2.10	0.017864	0.982136	0.964271
2.11	0.017429	0.982571	0.965142
2.12	0.017003	0.982997	0.965994
2.13	0.016586	0.983414	0.966829
2.14	0.016177	0.983823	0.967645
2.15	0.015778	0.984222	0.968445
2.16	0.015386	0.984614	0.969227
2.17	0.015003	0.984997	0.969993
2.18	0.014629	0.985371	0.970743
2.19	0.014262	0.985738	0.971476
2.20	0.013903	0.986097	0.972193
2.21	0.013553	0.986447	0.972895
2.22	0.013209	0.986791	0.973581
2.23	0.012874	0.987126	0.974253
2.24	0.012545	0.987455	0.974909
2.25	0.012224	0.987776	0.975551
2.26	0.011911	0.988089	0.976179
2.27	0.011604	0.988396	0.976792
2.28	0.011304	0.988696	0.977392
2.29	0.011011	0.988989	0.977979
2.30	0.010724	0.989276	0.978552
2.31	0.010444	0.989556	0.979112
2.32	0.010170	0.989830	0.979659
2.33	0.009903	0.990097	0.980194
2.34	0.009642	0.990358	0.980716
2.35	0.009387	0.990613	0.981227
2.36	0.009137	0.990863	0.981725
2.37	0.008894	0.991106	0.982212
2.38	0.008656	0.991344	0.982687
2.39	0.008424	0.991576	0.983152
2.40	0.008198	0.991802	0.983605
2.41	0.007976	0.992024	0.984047
2.42	0.007760	0.992240	0.984479
2.43	0.007549	0.992451	0.984901
2.44	0.007344	0.992656	0.985313
2.45	0.007143	0.992857	0.985714
2.46	0.006947	0.993053	0.986106
2.47	0.006756	0.993244	0.986489
2.48	0.006569	0.993431	0.986862
2.49	0.006387	0.993613	0.987226
2.50	0.006210	0.993790	0.987581

u	Φ (-u)	Φ (u)	D(u)
2.51	0.006037	0.993963	0.987927
2.52	0.005868	0.994132	0.988264
2.53	0.005703	0.994297	0.988594
2.54	0.005543	0.994457	0.988915
2.55	0.005386	0.994614	0.989228
2.56	0.005234	0.994766	0.989533
2.57	0.005085	0.994915	0.989830
2.58	0.004940	0.995060	0.990120
2.59	0.004799	0.995201	0.990402
2.60	0.004661	0.995339	0.990678
2.61	0.004527	0.995473	0.990946
2.62	0.004397	0.995603	0.991207
2.63	0.004269	0.995731	0.991461
2.64	0.004145	0.995855	0.991709
2.65	0.004025	0.995975	0.991951
2.66	0.003907	0.996093	0.992186
2.67	0.003793	0.996207	0.992415
2.68	0.003681	0.996319	0.992638
2.69	0.003573	0.996427	0.992855
2.70	0.003467	0.996533	0.993066
2.71	0.003364	0.996636	0.993272
2.72	0.003264	0.996736	0.993472
2.73	0.003167	0.996833	0.993666
2.74	0.003072	0.996928	0.993856
2.75	0.002980	0.997020	0.994040
2.76	0.002890	0.997110	0.994220
2.77	0.002803	0.997197	0.994394
2.78	0.002718	0.997282	0.994564
2.79	0.002635	0.997365	0.994729
2.80	0.002555	0.997445	0.994890
2.81	0.002477	0.997523	0.995046
2.82	0.002401	0.997599	0.995198
2.83	0.002327	0.997673	0.995345
2.84	0.002256	0.997744	0.995489
2.85	0.002186	0.997814	0.995628
2.86	0.002118	0.997882	0.995763
2.87	0.002052	0.997948	0.995895
2.88	0.001988	0.998012	0.996023
2.89	0.001926	0.998074	0.996147
2.90	0.001866	0.998134	0.996268
2.91	0.001807	0.998193	0.996386
2.92	0.001750	0.998250	0.996500
2.93	0.001695	0.998305	0.996610
2.94	0.001641	0.998359	0.996718
2.95	0.001589	0.998411	0.996822
2.96	0.001538	0.998462	0.996923
2.97	0.001489	0.998511	0.997022
2.98	0.001441	0.998559	0.997117
2.99	0.001395	0.998605	0.997210
3.00	0.001350	0.998650	0.997300

u	Φ(-u)	Φ(u)	D(u)
3.01	0.001306	0.998694	0.997387
3.02	0.001264	0.998736	0.997472
3.03	0.001223	0.998777	0.997554
3.04	0.001183	0.998817	0.997634
3.05	0.001144	0.998856	0.997711
3.06	0.001107	0.998893	0.997786
3.07	0.001070	0.998930	0.997859
3.08	0.001035	0.998965	0.997930
3.09	0.001001	0.998999	0.997998
3.10	0.000968	0.999032	0.998065
3.11	0.000936	0.999064	0.998129
3.12	0.000904	0.999096	0.998191
3.13	0.000874	0.999126	0.998252
3.14	0.000845	0.999155	0.998310
3.15	0.000816	0.999184	0.998367
3.16	0.000789	0.999211	0.998422
3.17	0.000762	0.999238	0.998475
3.18	0.000736	0.999264	0.998527
3.19	0.000711	0.999289	0.998577
3.20	0.000687	0.999313	0.998626
3.21	0.000664	0.999336	0.998673
3.22	0.000641	0.999359	0.998718
3.23	0.000619	0.999381	0.998762
3.24	0.000598	0.999402	0.998805
3.25	0.000577	0.999423	0.998846
3.26	0.000557	0.999443	0.998886
3.27	0.000538	0.999462	0.998924
3.28	0.000519	0.999481	0.998962
3.29	0.000501	0.999499	0.998998
3.30	0.000483	0.999517	0.999033
3.31	0.000467	0.999533	0.999067
3.32	0.000450	0.999550	0.999100
3.33	0.000434	0.999566	0.999131
3.34	0.000419	0.999581	0.999162
3.35	0.000404	0.999596	0.999192
3.36	0.000390	0.999610	0.999220
3.37	0.000376	0.999624	0.999248
3.38	0.000362	0.999638	0.999275
3.39	0.000350	0.999650	0.999301
3.40	0.000337	0.999663	0.999326
3.41	0.000325	0.999675	0.999350
3.42	0.000313	0.999687	0.999374
3.43	0.000302	0.999698	0.999396
3.44	0.000291	0.999709	0.999418
3.45	0.000280	0.999720	0.999439
3.46	0.000270	0.999730	0.999460
3.47	0.000260	0.999740	0.999479
3.48	0.000251	0.999749	0.999498
3.49	0.000242	0.999758	0.999517
3.50	0.000233	0.999767	0.999535
3.51	0.000224	0.999776	0.999552
3.52	0.000216	0.999784	0.999568
3.53	0.000208	0.999792	0.999584
3.54	0.000200	0.999800	0.999600
3.55	0.000193	0.999807	0.999615
3.56	0.000185	0.999815	0.999629
3.57	0.000179	0.999821	0.999643
3.58	0.000172	0.999828	0.999656
3.59	0.000165	0.999835	0.999669
3.60	0.000159	0.999841	0.999682
3.61	0.000153	0.999847	0.999694
3.62	0.000147	0.999853	0.999705
3.63	0.000142	0.999858	0.999717
3.64	0.000136	0.999864	0.999727
3.65	0.000131	0.999869	0.999738
3.66	0.000126	0.999874	0.999748
3.67	0.000121	0.999879	0.999757
3.68	0.000117	0.999883	0.999767
3.69	0.000112	0.999888	0.999776
3.70	0.000108	0.999892	0.999784
3.71	0.000104	0.999896	0.999793
3.72	0.000100	0.999900	0.999801
3.73	0.000096	0.999904	0.999808
3.74	0.000092	0.999908	0.999816
3.75	0.000088	0.999912	0.999823
3.76	0.000085	0.999915	0.999830
3.77	0.000082	0.999918	0.999837
3.78	0.000078	0.999922	0.999843
3.79	0.000075	0.999925	0.999849
3.80	0.000072	0.999928	0.999855
3.81	0.000070	0.999930	0.999861
3.82	0.000067	0.999933	0.999867
3.83	0.000064	0.999936	0.999872
3.84	0.000062	0.999938	0.999877
3.85	0.000059	0.999941	0.999882
3.86	0.000057	0.999943	0.999887
3.87	0.000054	0.999946	0.999891
3.88	0.000052	0.999948	0.999896
3.89	0.000050	0.999950	0.999900
3.90	0.000048	0.999952	0.999904
3.91	0.000046	0.999954	0.999908
3.92	0.000044	0.999956	0.999911
3.93	0.000042	0.999958	0.999915
3.94	0.000041	0.999959	0.999918
3.95	0.000039	0.999961	0.999922
3.96	0.000037	0.999963	0.999925
3.97	0.000036	0.999964	0.999928
3.98	0.000034	0.999966	0.999931
3.99	0.000033	0.999967	0.999934
4.00	0.000032	0.999968	0.999937

Table 5

Factors k for the calculation of limiting values

	Probability P	
n	95 %	99 %
2	19.2	99
3	5.1	11.6
4	3.3	5.8
5	2.6	4.0
6	2.2	3.2
7	2.05	2.8
8	1.90	2.5
9	1.80	2.3
10	1.72	2.2
12	1.62	2.0
14	1.54	1.85
16	1.48	1.75
18	1.44	1.68
20	1.40	1.62
25	1.34	1.52
30	1.30	1.45
35	1.27	1.40
40	1.25	1.37
50	1.22	1.32
60	1.19	1.28
120	1.12	1.18

Table 6

Quantiles of the t distribution (two-sided)

	Probability		
f	95 %	99 %	99.9 %
1	12.7	63.7	636.6
2	4.3	9.93	31.6
3	3.18	5.84	12.9
4	2.78	4.60	8.61
5	2.57	4.03	6.87
6	2.45	3.71	5.96
7	2.37	3.50	5.41
8	2.31	3.36	5.04
9	2.26	3.25	4.78
10	2.23	3.17	4.59
11	2.20	3.11	4.44
12	2.18	3.06	4.32
13	2.16	3.01	4.22
14	2.15	2.98	4.14
15	2.13	2.95	4.07
16	2.12	2.92	4.02
17	2.11	2.90	3.97
18	2.10	2.88	3.92
19	2.09	2.86	3.88
20	2.09	2.85	3.85
25	2.06	2.79	3.73
30	2.04	2.75	3.65
35	2.03	2.72	3.59
40	2.02	2.70	3.55
45	2.01	2.69	3.52
50	2.01	2.68	3.50
100	1.98	2.63	3.39
200	1.97	2.60	3.34
300	1.97	2.59	3.32
400	1.97	2.59	3.32
500	1.97	2.59	3.31
∞	1.96	2.58	3.30

Table 7

Factors for the determination of the confidence interval of a standard deviation

	P = 95 %		P = 99 %	
n	c_1	c_2	c_1	c_2
2	0.45	32.3	0.36	167
3	0.52	6.29	0.43	14.1
4	0.57	3.73	0.48	6.45
5	0.60	2.87	0.52	4.41
6	0.62	2.45	0.55	3.48
7	0.64	2.20	0.57	2.98
8	0.66	2.04	0.59	2.66
9	0.68	1.92	0.60	2.44
10	0.69	1.83	0.62	2.28
15	0.73	1.58	0.67	1.86
20	0.76	1.46	0.70	1.67
25	0.78	1.39	0.72	1.56
30	0.80	1.35	0.75	1.49
40	0.82	1.28	0.77	1.40
50	0.83	1.25	0.79	1.34
100	0.88	1.16	0.84	1.22

Table 8 Quantiles of the F distribution (one-sided to the value 95 %)

f_2	$f_1=1$	$f_1=2$	$f_1=3$	$f_1=4$	$f_1=5$	$f_1=6$	$f_1=7$	$f_1=8$	$f_1=9$
1	161	200	216	225	230	234	237	239	241
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
32	4.15	3.30	2.90	2.67	2.51	2.40	2.31	2.24	2.19
34	4.13	3.28	2.88	2.65	2.49	2.38	2.29	2.23	2.17
36	4.11	3.26	2.87	2.63	2.48	2.36	2.28	2.21	2.15
38	4.10	3.24	2.85	2.62	2.46	2.35	2.26	2.19	2.14
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
70	3.98	3.13	2.74	2.50	2.35	2.23	2.14	2.07	2.02
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00
90	3.95	3.10	2.71	2.47	2.32	2.20	2.11	2.04	1.99
100	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97
150	3.90	3.06	2.66	2.43	2.27	2.16	2.07	2.00	1.94
200	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93
1000	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89

Table 8 (cont.) Quantiles of the F distribution (one-sided to the value 95 %)

f_2	$f_1 = 10$	$f_1 = 15$	$f_1 = 20$	$f_1 = 30$	$f_1 = 40$	$f_1 = 50$	$f_1 = 100$	$f_1 \to \infty$
1	242	246	248	250	251	252	253	254
2	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5
3	8.79	8.70	8.66	8.62	8.59	8.58	8.55	8.53
4	5.96	5.86	5.80	5.75	5.72	5.70	5.66	5.63
5	4.74	4.62	4.56	4.50	4.46	4.44	4.41	4.37
6	4.06	3.94	3.87	3.81	3.77	3.75	3.71	3.67
7	3.64	3.51	3.44	3.38	3.34	3.32	3.27	3.23
8	3.35	3.22	3.15	3.08	3.04	3.02	2.97	2.93
9	3.14	3.01	2.94	2.86	2.83	2.80	2.76	2.71
10	2.98	2.85	2.77	2.70	2.66	2.64	2.59	2.54
11	2.85	2.72	2.65	2.57	2.53	2.51	2.46	2.40
12	2.75	2.62	2.54	2.47	2.43	2.40	2.35	2.30
13	2.67	2.53	2.46	2.38	2.34	2.31	2.26	2.21
14	2.60	2.46	2.39	2.31	2.27	2.24	2.19	2.13
15	2.54	2.40	2.33	2.25	2.20	2.18	2.12	2.07
16	2.49	2.35	2.28	2.19	2.15	2.12	2.07	2.01
17	2.45	2.31	2.23	2.15	2.10	2.08	2.02	1.96
18	2.41	2.27	2.19	2.11	2.06	2.04	1.98	1.92
19	2.38	2.23	2.16	2.07	2.03	2.00	1.94	1.88
20	2.35	2.20	2.12	2.04	1.99	1.97	1.91	1.84
22	2.30	2.15	2.07	1.98	1.94	1.91	1.85	1.78
24	2.25	2.11	2.03	1.94	1.89	1.86	1.80	1.73
26	2.22	2.07	1.99	1.90	1.85	1.82	1.76	1.69
28	2.19	2.04	1.96	1.87	1.82	1.79	1.73	1.65
30	2.16	2.01	1.93	1.84	1.79	1.76	1.70	1.62
32	2.14	1.99	1.91	1.82	1.77	1.74	1.67	1.59
34	2.12	1.97	1.89	1.80	1.75	1.71	1.65	1.57
36	2.11	1.95	1.87	1.78	1.73	1.69	1.62	1.55
38	2.09	1.94	1.85	1.76	1.71	1.68	1.61	1.53
40	2.08	1.92	1.84	1.74	1.69	1.66	1.59	1.51
50	2.03	1.87	1.78	1.69	1.63	1.60	1.52	1.44
60	1.99	1.84	1.75	1.65	1.59	1.56	1.48	1.39
70	1.97	1.81	1.72	1.62	1.57	1.53	1.45	1.35
80	1.95	1.79	1.70	1.60	1.54	1.51	1.43	1.32
90	1.94	1.78	1.69	1.59	1.53	1.49	1.41	1.30
100	1.93	1.77	1.68	1.57	1.52	1.48	1.39	1.28
150	1.89	1.73	1.64	1.53	1.48	1.44	1.34	1.22
200	1.88	1.72	1.62	1.52	1.46	1.41	1.32	1.19
1000	1.84	1.68	1.58	1.47	1.41	1.36	1.26	1.08

Table 8 Qantiles of the F distribution (one-sided to the value 99 %)

f_2	$f_1=1$	$f_1=2$	$f_1=3$	$f_1=4$	$f_1=5$	$f_1=6$	$f_1=7$	$f_1=8$	$f_1=9$
1	4052	4999	5403	5625	5764	5859	5928	5982	6022
2	98.5	99.0	99.2	99.3	99.3	99.3	99.4	99.4	99.4
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
32	7.50	5.34	4.46	3.97	3.65	3.43	3.26	3.13	3.02
34	7.44	5.29	4.42	3.93	3.61	3.39	3.22	3.09	2.98
36	7.40	5.25	4.38	3.89	3.57	3.35	3.18	3.05	2.95
38	7.35	5.21	4.34	3.86	3.54	3.32	3.15	3.02	2.92
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.79
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
70	7.01	4.92	4.08	3.60	3.29	3.07	2.91	2.78	2.67
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64
90	6.93	4.85	4.01	3.54	3.23	3.01	2.84	2.72	2.61
100	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59
150	6.81	4.75	3.92	3.45	3.14	2.92	2.76	2.63	2.53
200	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50
1000	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.43

Table 8 (cont.) Quantiles of the F distribution (one-sided to the value 99 %)

f_2	$f_1 = 10$	$f_1 = 15$	$f_1 = 20$	$f_1 = 30$	$f_1 = 40$	$f_1 = 50$	$f_1 = 100$	$f_1 \to \infty$
1	6056	6157	6209	6261	6287	6300	6330	6366
2	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5
3	27.2	26.9	26.7	26.5	26.4	26.4	26.2	26.1
4	14.5	14.2	14.0	13.8	13.7	13.7	13.6	13.5
5	10.1	9.72	9.55	9.38	9.29	9.24	9.13	9.02
6	7.87	7.56	7.40	7.23	7.14	7.09	6.99	6.88
7	6.62	6.31	6.16	5.99	5.91	5.86	5.75	5.65
8	5.81	5.52	5.36	5.20	5.12	5.07	4.96	4.86
9	5.26	4.96	4.81	4.65	4.57	4.52	4.42	4.31
10	4.85	4.56	4.41	4.25	4.17	4.12	4.01	3.91
11	4.54	4.25	4.10	3.94	3.86	3.81	3.71	3.60
12	4.30	4.01	3.86	3.70	3.62	3.57	3.47	3.36
13	4.10	3.82	3.66	3.51	3.43	3.38	3.27	3.17
14	3.94	3.66	3.51	3.35	3.27	3.22	3.11	3.00
15	3.80	3.52	3.37	3.21	3.13	3.08	2.98	2.87
16	3.69	3.41	3.26	3.10	3.02	2.97	2.86	2.75
17	3.59	3.31	3.16	3.00	2.92	2.87	2.76	2.65
18	3.51	3.23	3.08	2.92	2.84	2.78	2.68	2.57
19	3.43	3.15	3.00	2.84	2.76	2.71	2.60	2.49
20	3.37	3.09	2.94	2.78	2.69	2.64	2.54	2.42
22	3.26	2.98	2.83	2.67	2.58	2.53	2.42	2.31
24	3.17	2.89	2.74	2.58	2.49	2.44	2.33	2.21
26	3.09	2.82	2.66	2.50	2.42	2.36	2.25	2.13
28	3.03	2.75	2.60	2.44	2.35	2.30	2.19	2.06
30	2.98	2.70	2.55	2.39	2.30	2.25	2.13	2.01
32	2.93	2.66	2.50	2.34	2.25	2.20	2.08	1.96
34	2.89	2.62	2.46	2.30	2.21	2.16	2.04	1.91
36	2.86	2.58	2.43	2.26	2.17	2.12	2.00	1.87
38	2.83	2.55	2.40	2.23	2.14	2.09	1.97	1.84
40	2.80	2.52	2.37	2.20	2.11	2.06	1.94	1.80
50	2.70	2.42	2.27	2.10	2.01	1.95	1.82	1.68
60	2.63	2.35	2.20	2.03	1.94	1.88	1.75	1.60
70	2.59	2.31	2.15	1.98	1.89	1.83	1.70	1.54
80	2.55	2.27	2.12	1.94	1.85	1.79	1.66	1.49
90	2.52	2.24	2.09	1.92	1.82	1.76	1.62	1.46
100	2.50	2.22	2.07	1.89	1.80	1.73	1.60	1.43
150	2.44	2.16	2.00	1.83	1.73	1.66	1.52	1.33
200	2.41	2.13	1.97	1.79	1.69	1.63	1.48	1.28
1000	2.34	2.06	1.90	1.72	1.61	1.54	1.38	1.11

Table 9 Quantiles of the F distribution (two-sided to the value 95 %)

f_2	$f_1=1$	$f_1=2$	$f_1=3$	$f_1=4$	$f_1=5$	$f_1=6$	$f_1=7$	$f_1=8$	$f_1=9$
1	648	800	864	900	922	937	948	957	963
2	38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4
3	17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5
4	12.2	10.6	9.98	9.60	9.36	9.20	9.07	8.98	8.90
5	10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78
11	6.72	5.25	4.63	4.27	4.04	3.88	3.76	3.66	3.59
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45
50	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33
70	5.25	3.89	3.31	2.97	2.75	2.59	2.47	2.38	2.30
80	5.22	3.86	3.28	2.95	2.73	2.57	2.45	2.35	2.28
90	5.20	3.84	3.26	2.93	2.71	2.55	2.43	2.34	2.26
100	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24
200	5.09	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18
500	5.05	3.72	3.14	2.81	2.59	2.43	2.31	2.22	2.14

Table 9 (cont.) Quantiles of the F distribution (two-sided to the value 95 %)

f_2	$f_1 = 10$	$f_1 = 15$	$f_1 = 20$	$f_1 = 30$	$f_1 = 40$	$f_1 = 50$	$f_1 = 100$	$f_1 \to \infty$
1	969	985	993	1001	1006	1008	1013	1018
2	39.4	39.4	39.4	39.5	39.5	39.5	39.5	39.5
3	14.4	14.3	14.2	14.1	14.0	14.0	14.0	13.9
4	8.84	8.66	8.56	8.46	8.41	8.38	8.32	8.26
5	6.62	6.43	6.33	6.23	6.18	6.14	6.08	6.02
6	5.46	5.27	5.17	5.07	5.01	4.98	4.92	4.85
7	4.76	4.57	4.47	4.36	4.31	4.28	4.21	4.14
8	4.30	4.10	4.00	3.89	3.84	3.81	3.74	3.67
9	3.96	3.77	3.67	3.56	3.51	3.47	3.40	3.33
10	3.72	3.52	3.42	3.31	3.26	3.22	3.15	3.08
11	3.52	3.34	3.22	3.12	3.06	3.03	2.95	2.88
12	3.37	3.18	3.07	2.96	2.91	2.87	2.80	2.72
13	3.25	3.05	2.95	2.84	2.78	2.74	2.67	2.60
14	3.15	2.95	2.84	2.73	2.67	2.64	2.56	2.49
15	3.06	2.86	2.76	2.64	2.58	2.55	2.47	2.40
16	2.99	2.79	2.68	2.57	2.51	2.47	2.40	2.32
17	2.92	2.72	2.62	2.50	2.44	2.41	2.33	2.25
18	2.87	2.67	2.56	2.44	2.38	2.35	2.27	2.19
19	2.82	2.62	2.51	2.39	2.33	2.30	2.22	2.13
20	2.77	2.57	2.46	2.35	2.29	2.25	2.17	2.09
22	2.70	2.50	2.39	2.27	2.21	2.17	2.09	2.00
24	2.64	2.44	2.33	2.21	2.15	2.11	2.02	1.94
26	2.59	2.39	2.28	2.16	2.09	2.05	1.97	1.88
28	2.55	2.34	2.23	2.11	2.05	2.01	1.92	1.83
30	2.51	2.31	2.20	2.07	2.01	1.97	1.88	1.79
40	2.39	2.18	2.07	1.94	1.88	1.83	1.74	1.64
50	2.32	2.11	1.99	1.87	1.80	1.75	1.66	1.55
60	2.27	2.06	1.94	1.82	1.74	1.70	1.60	1.48
70	2.24	2.03	1.91	1.78	1.71	1.66	1.56	1.44
80	2.21	2.00	1.88	1.75	1.68	1.63	1.53	1.40
90	2.19	1.98	1.86	1.73	1.66	1.61	1.50	1.37
100	2.18	1.97	1.85	1.71	1.64	1.59	1.48	1.35
200	2.11	1.90	1.78	1.64	1.56	1.51	1.39	1.23
500	2.07	1.86	1.74	1.60	1.51	1.46	1.34	1.14

Table 9 Quantiles of the F distribution (two-sided to the value 99 %)

f_2	$f_1 = 1$	$f_1 = 2$	$f_1 = 3$	$f_1 = 4$	$f_1 = 5$	$f_1 = 6$	$f_1 = 7$	$f_1 = 8$	$f_1 = 9$
1	16200	20000	21600	22500	23100	23400	23700	23900	24100
2	198	199	199	199	199	199	199	199	199
3	55.6	49.8	47.4	46.2	45.3	44.8	44.4	44.1	43.8
4	31.3	26.3	24.3	23.2	22.5	22.0	21.6	21.4	21.1
5	22.8	18.3	16.5	15.6	14.9	14.5	14.2	14.0	13.8
6	18.6	14.5	12.9	12.0	11.5	11.1	10.8	10.6	10.4
7	16.2	12.4	10.9	10.1	9.52	9.16	8.89	8.68	8.51
8	14.7	11.0	9.60	8.80	8.30	7.95	7.69	7.50	7.34
9	13.6	10.1	8.72	7.96	7.47	7.13	6.89	6.69	6.54
10	12.8	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97
11	12.2	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54
12	11.8	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20
13	11.4	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.93
14	11.1	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72
15	10.8	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54
16	10.6	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38
17	10.4	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25
18	10.2	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14
19	10.1	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22
50	8.63	5.90	4.83	4.23	3.85	3.58	3.38	3.22	3.09
60	8.49	5.80	4.73	4.14	3.76	3.49	3.29	3.13	3.01
70	8.40	5.72	4.66	4.08	3.70	3.43	3.23	3.08	2.95
80	8.33	5.67	4.61	4.03	3.65	3.39	3.19	3.03	2.91
90	8.28	5.62	4.57	3.99	3.62	3.35	3.15	3.00	2.87
100	8.24	5.59	4.54	3.96	3.59	3.33	3.13	2.97	2.85
200	8.06	5.44	4.40	3.84	3.47	3.21	3.01	2.86	2.73
500	7.95	5.36	4.33	3.76	3.40	3.14	2.94	2.79	2.66

Table 9 (cont.) Quantiles of the F distribution (two-sided to the value 99 %)

f_2	$f_1 = 10$	$f_1 = 15$	$f_1 = 20$	$f_1 = 30$	$f_1 = 40$	$f_1 = 50$	$f_1 = 100$	$f_1 \to \infty$
1	24200	24600	24800	25000	25100	25200	25300	25500
2	199	199	199	199	199	199	199	200
3	43.7	43.1	42.8	42.5	42.4	42.2	42.0	41.8
4	21.0	20.4	20.2	19.9	19.8	19.7	19.5	19.3
5	13.6	13.1	12.9	12.7	12.5	12.5	12.3	12.1
6	10.3	9.81	9.59	9.36	9.24	9.17	9.03	8.88
7	8.38	7.97	7.75	7.53	7.42	7.35	7.22	7.08
8	7.21	6.81	6.61	6.40	6.29	6.22	6.09	5.95
9	6.42	6.03	5.83	5.62	5.52	5.45	5.32	5.19
10	5.85	5.47	5.27	5.07	4.97	4.90	4.77	4.64
11	5.42	5.05	4.86	4.65	4.55	4.49	4.36	4.23
12	5.09	4.72	4.53	4.33	4.23	4.17	4.04	3.90
13	4.82	4.46	4.27	4.07	3.97	3.91	3.78	3.65
14	4.60	4.25	4.06	3.86	3.76	3.70	3.57	3.44
15	4.42	4.07	3.88	3.69	3.58	3.52	3.39	3.26
16	4.27	3.92	3.73	3.54	3.44	3.37	3.25	3.11
17	4.14	3.79	3.61	3.41	3.31	3.25	3.12	2.98
18	4.03	3.68	3.50	3.30	3.20	3.14	3.01	2.87
19	3.93	3.59	3.40	3.21	3.11	3.04	2.91	2.78
20	3.85	3.50	3.32	3.12	3.02	2.96	2.83	2.69
22	3.70	3.36	3.18	2.98	2.88	2.82	2.69	2.55
24	3.59	3.25	3.06	2.87	2.77	2.70	2.57	2.43
26	3.49	3.15	2.97	2.77	2.67	2.61	2.47	2.33
28	3.41	3.07	2.89	2.69	2.59	2.53	2.39	2.25
30	3.34	3.01	2.82	2.63	2.52	2.46	2.32	2.18
40	3.12	2.78	2.60	2.40	2.30	2.23	2.09	1.93
50	2.99	2.65	2.47	2.27	2.16	2.10	1.95	1.79
60	2.90	2.57	2.39	2.19	2.08	2.01	1.86	1.69
70	2.85	2.51	2.33	2.13	2.02	1.95	1.80	1.62
80	2.80	2.47	2.29	2.08	1.97	1.90	1.75	1.56
90	2.77	2.44	2.25	2.05	1.94	1.87	1.71	1.52
100	2.74	2.41	2.23	2.02	1.91	1.84	1.68	1.49
200	2.63	2.30	2.11	1.91	1.79	1.71	1.54	1.31
500	2.56	2.23	2.04	1.84	1.72	1.64	1.46	1.18

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Robert Bosch GmbH
C/QMM
Postfach 30 02 20
D-70442 Stuttgart
Germany
Phone +49 711 811 - 0
www.bosch.com

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C/QMM
Postfach 30 02 20
D-70442 Stuttgart
Germany
Phone +49 711 811-0
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