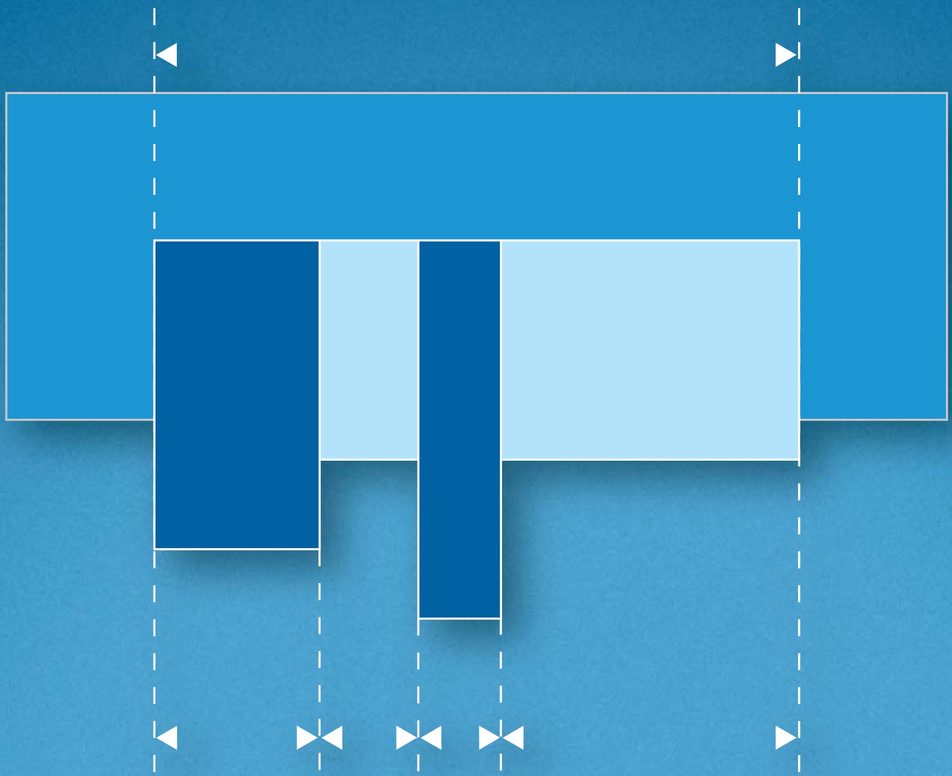


Quality Management in the Bosch Group | Technical Statistics

5. Statistical **Tolerancing**

BOSCH

Invented for life

Quality Management in the Bosch Group
Technical Statistics

Booklet No. 5

Statistical Tolerancing

Edition 02.2022

3rd Edition, February 2022

2nd Edition 05.2013

1st Edition 09.1993

Contents

1. Introduction

Statistical Tolerancing is a method to determine tolerances on the basis of statistical principles. The purpose is to determine a tolerance for the assembly characteristic, which is a combination of several individual characteristics.

Statistical Tolerancing is based on the idea that random positive and negative deviations of actual values of individual parts characteristics from their respective nominal values will neutralize each other as a rule, when these individual parts are "stacked" in an assembly.

When the tolerance for an assembly characteristic is given, then, under the aforesaid consideration, greater tolerances result for the individual characteristics than those resulting from mere arithmetical calculation. The resultant theoretical advantages with respect to the production technique and production costs have been explained in detail in many papers and publications on this subject. The fundamentals of this method were known long back. Its development can be traced back to the twenties of last century.

Within the scope of this booklet a short summary will be provided on the theoretical background of Statistical Tolerancing, the possibilities for application will be set out, and the necessary constraints described.

2. Fundamentals

2.1. Dimension Chain

The assembly of individual parts within a technical system (an assembly) leads often to the situation, that independent dimensions of individual parts form an interrelated chain of linear dimensions, viewed purely from geometric aspects, which in its totality becomes the so-called “assembly dimension”. It is called, suggestively, a dimension chain, consisting of many individual dimensions.

The dimension chain is called linear, if all the individual dimensions can be presented through arrows, parallel to each other, which form a closed chain of lines. The direction of the arrows denotes the positive or negative contributions of the individual dimensions.

A dimension is called positive (negative), if, with the change of its dimension under the restriction, that all other dimensions of the chain remaining constant, the assembly dimension changes in the same (opposite) sense.

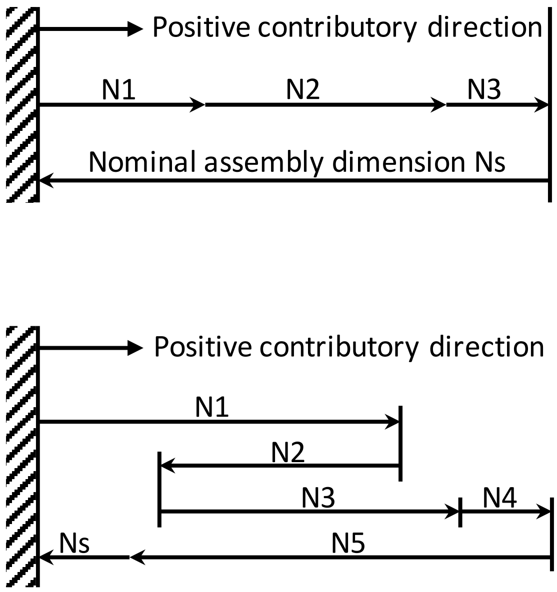


Fig. 2.1: Dimension chains consisting of individual nominal dimensions N_i. Top: only positive contributory direction. Bottom: positive and negative contributory direction.

Apart from this there are plane (two-dimensional) and spatial (three-dimensional) dimension chains. Plane (two-dimensional) dimension chains often result from interaction of rotating parts or movement of rocker arms or cams, where angles and radii play a part. In such nonlinear dimension chains complicated correlations of individual dimensions can occur, which can however be presented in most of the cases through analytical functions.

A simple example of a (two-element) nonlinear dimension chain is that of two holes, whose position on two axes perpendicular to each other is given by the respective distance from a reference point (intersection point of the axes).

The relative distance of the two holes is then defined by

$$z = \sqrt{x^2 + y^2} \ . \qquad (2.1)$$

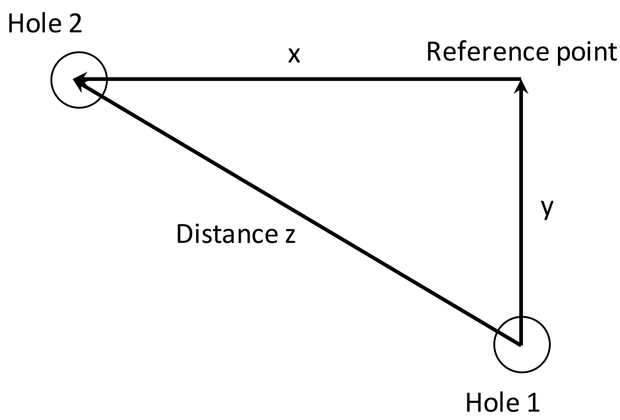


Fig. 2.2: Example of a two-element nonlinear dimension chain.

Tolerances are determined according to DIN ISO 286 [4] by indicating a maximum dimension G_o and a minimum dimension G_u:

$$T = G_o - G_u . \tag{2.2}$$

The midpoint of the tolerance is

$$C = \frac{G_o + G_u}{2} . \tag{2.3}$$

If the nominal dimension of a characteristic does not coincide with this midpoint, then the tolerance is specified by way of a lower and upper deviation to the nominal dimension (difference between minimum and nominal dimension and between maximum and nominal dimension).

Historically, toleration has been introduced first with respect to geometrical characteristics like length and diameter. They are called, suggestively, dimensions (nominal dimension, actual dimension, deviation to the nominal dimension etc.). But toleration can be used without difficulty also with respect to non-geometrical characteristics like resistors. This is the reason that in the following the general term "characteristic" will be preferred over "dimension".

2.2. Gaussian Law of Error Propagation

The Gaussian law of error propagation describes how the measurement errors of several uncorrelated characteristics x_i affect an assembly characteristic z, which is calculated in accordance with a function

$$z = f\left(x_1, x_2, \ldots, x_k\right), \tag{2.4}$$

so that [1]:

$$s_z^2 \approx \left(\frac{\partial f}{\partial x_1}\right)^2 s_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 s_2^2 + \ldots + \left(\frac{\partial f}{\partial x_k}\right)^2 s_k^2 . \tag{2.5}$$

Z is, then, only an indirectly measurable quantity. For example, the area of a rectangle is determined by measuring the length of sides and multiplying the results of measurement with each other.

The accuracy with which z can be indicated depends on the accuracy of the individual characteristics x_i. In general, for each respective x_i the average $\overline{x}_i$ and the standard deviation s_i of a sequence of measurements are given.

In the above expression, $\partial z / \partial x_i$ denotes the partial derivatives of the function z with respect to the characteristics x_i. They are to be calculated at $\overline{x}_i$.

The derivation of the formula for s_z includes the expansion of z in a Taylor's series and overlooks the terms of higher order. Because of this reason, instead of the equals sign the wave symbol is used in the formula, indicating that the two terms are approximately equal.

The application of the law of error propagation is understood easily with an example.

EXAMPLE:

We consider two Ohmic resistors, on which a series of measurements were carried out. As a result of this measurement series, the mean and the standard deviation are given in each case, the latter being a measure for the average error:

$R_1 = 47 \pm 0.8\Omega$, $R_2 = 68 \pm 1.1\Omega$.

In the event of parallel connection, the total resistance R

$R = \frac{R_1 \cdot R_2}{R_1 + R_2}$ *is calculated as follows:* $R = \frac{47 \cdot 68}{47+68} = 27.8\Omega$.

To calculate the corresponding average error, first it is necessary to determine the partial derivatives of R with respect to R_1 and R_2:

$$\frac{\partial R}{\partial R_1} = \frac{R_2^2}{(R_1 + R_2)^2} = \frac{68^2}{(47+68)^2} = 0.35,\ \frac{\partial R}{\partial R_2} = \frac{R_1^2}{(R_1 + R_2)^2} = \frac{47^2}{(47+68)^2} = 0.167.$$

The average error of the total resistance is obtained by substituting these expressions in the law of error propagation:

$$s_R^2 = 0.35^2 \cdot 0.8^2 + 0.167^2 \cdot 1.1^2 = 0.11 \ \Rightarrow s_R = 0.33.$$

Accordingly, the result is:

$R = 27.8 \pm 0.3\Omega$.

The law of error propagation becomes simple to present, when the function *f*, which defines the link of the independent characteristics x_i is a sum. Then the partial derivatives are all equal to one, and we find:

$$s_z^2 = s_1^2 + s_2^2 + ... + s_k^2. \quad (2.6)$$

Accordingly, for the variances of the respective underlying populations the following is valid:

$$\sigma_z^2 = \sigma_1^2 + \sigma_2^2 + ... + \sigma_k^2. \quad (2.7)$$

This means, the variance of an assembly characteristic, which is determined by addition of independent individual characteristics, is equal to the sum of the variances of these individual characteristics.

EXAMPLE:

If the two resistors are connected in series, we find:

$s_R^2 = s_{R_1}^2 + s_{R_2}^2 = (0.8\,\Omega)^2 + (1.1\,\Omega)^2 \ \Rightarrow s_R = 1.36\Omega$, *thus:* $R = 115 \pm 1.4\Omega$.

2.3. Central Limit Theorem of Statistics

The Central Limit Theorem of Statistics states that due to random interaction of many individual characteristics (addition of random variables) the assembly characteristic obtained is approximately normally distributed.

With this abstraction, which includes consideration of a characteristic of a part as a random variable, the phenomenon of deviation of actual characteristic value from the desired nominal characteristic value, as experienced in practice in production, can be described mathematically. The fact defined by the Central Limit Theorem of Statistics may be explained by a simple example.

EXAMPLE:

The result of a throw with a regular dice is a random number x, which can assume the values 1, 2, 3, 4, 5 and 6. The probability that a "6" is thrown is 1/6. The same applies to all other values. The respective probability function f(x) is presented in Figure 2.3. Pure mathematical calculation results in the value of 3.5 as the average of many throws.

If the total score from a throw of two dice is the considered random variable, the representation of the probability function of this random variable shows a "triangular form".

The least possible result is 2, the maximum possible result is 12. The probability of getting a total score of 7 is obtained by dividing the number of all combinations yielding a total score of 7 by the number of all possible combinations (36).

There are 6 possibilities to get the total score 7 and they are: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1), and in all 36 possible results from the random process considered here: (1,1), (1,2), ..., (1,6), (2,1), (2,2), ..., (2,6), ..., (6,6). Thus it is: f(7) = 6/36 = 1/6.

The random variable "total score as results from a throw with 4 dice" has a probability density, whose representation is already very similar to the curve depicting the probability density of the normal distribution (continuous line).

*The mean of this random variable is 4*3.5 = 14. Thus, we obtain, as random result, values which are often very close to this mean and only occasionally big deviations from the mean, for example, the value 24, which corresponds to the throw result (6, 6, 6, 6).*

Said otherwise, to arrive at this result each of the four random variables (dice) must assume a value (6) which differs very much from the mean 3.5. The probability for this result is very small: $(1/6)^4 < 1$ *%.*

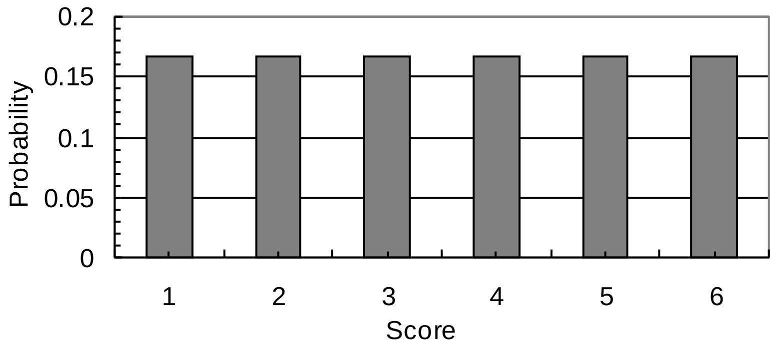


Fig. 2.3: Probability function of a regular dice.

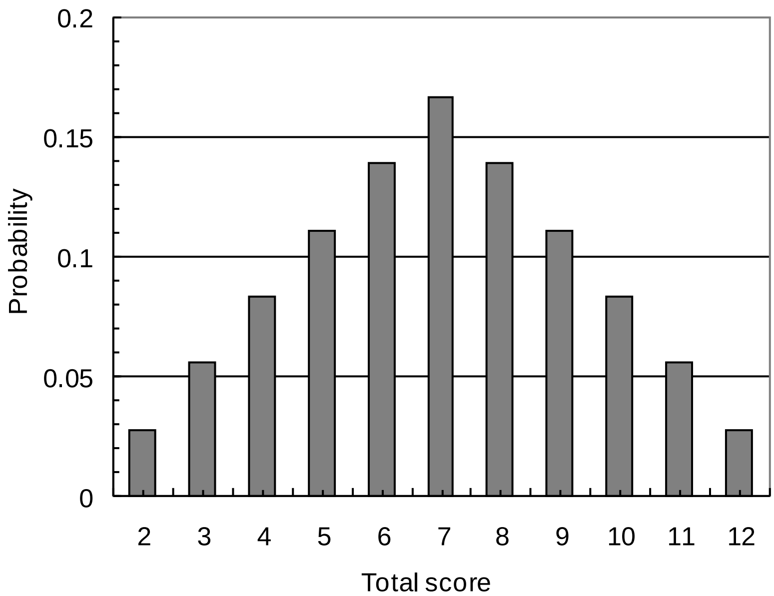


Fig. 2.4: Probability function of the random variable: "Total score from a throw with two dice"

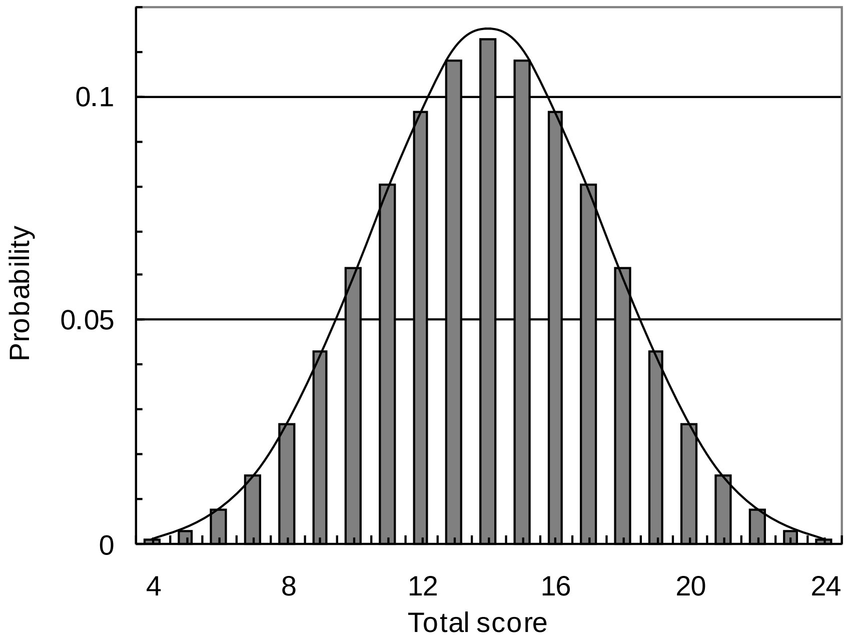


Fig. 2.5: Probability function of the random variable: "Total score from a throw with four dice"

This principle is of basic importance for the theory of Statistical Tolerancing. Statistical Tolerancing makes use of the fact, that in a random combination of a sufficiently big number of individual characteristics positive and negative characteristic deviations neutralize each other.

The result from the example shown above can be applied suitably to the properties of a dimension chain. In order that the assembly dimension of *k* parts assumes a value which deviates very much from the nominal assembly dimension, actual dimension values of all *k* individual parts must simultaneously be at their respective tolerance limit. The probability for this situation is however very small for a sufficiently big number of *k*. Assuming a random selection of parts in assembly, the distribution of the random variable “assembly characteristic” is approximately a normal distribution.

The following diagram should make this point clear.

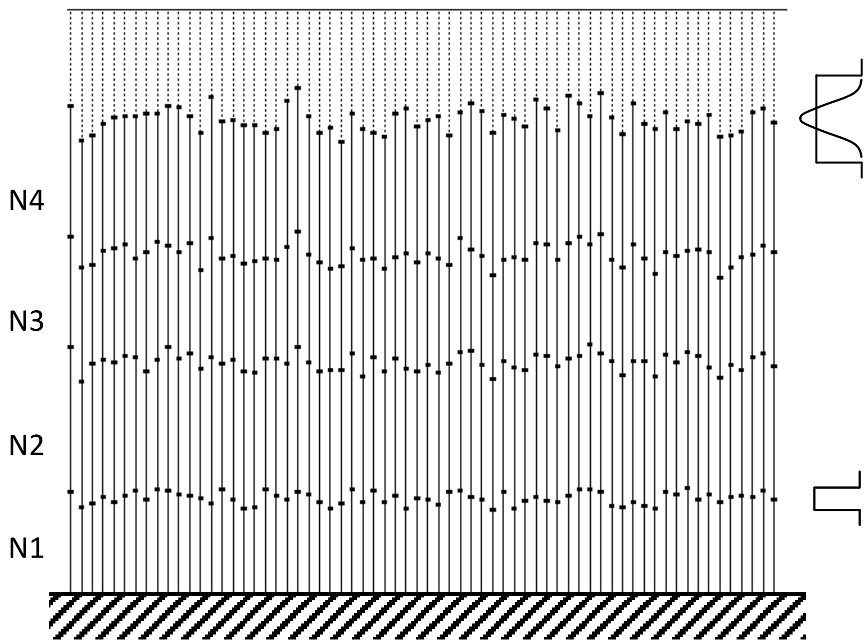


Fig. 2.6: Schematic representation of an assembly consisting of 4 parts whose individual characteristics are subject to a rectangular distribution. All 66 assemblies are arranged in sequence. The assembly characteristic which is referred to the top edge (broken line) is approximately normally distributed.

The figure shows schematically the interaction of four individual dimensions, having identical tolerance and following rectangular distribution, for an example of 66 assemblies in all, which are arranged here in sequence. It is seen that the assembly dimension (broken line) N_s which is referred to the top edge shows big deviations from the average value only seldom. The range $\mu \pm 3 \cdot \sigma$ of the normal distribution, approximately resulting for the assembly dimension, is obviously somewhat smaller than the arithmetically calculated tolerance, which is shown by the width of big rectangle.

The randomness of combination of parts is only one of the many conditions which must be satisfied so that this statistical behavior may be used advantageously for determining tolerances.

> *REMARKS:*
>
> *Based on the dice example, it was shown as to how the probability function of a sum of random variables is determined, which can only assume values of whole numbers (1, ..., 6). This mathematical operation is also termed "convolution". The "convolution" thus denotes the process of linking the probability functions of random variables. This can be applied also to continuous variables, i.e., such variables which can assume any real value.*
>
> *Linear characteristics of parts can be considered as continuous variables because they can assume any interim value on a scale. In the case of measuring instruments with analog display, such as slide caliper rule, micrometer screw or indicating precision gauge, this situation is obvious, in contrast to those with digital display. Strictly speaking, a linear characteristic can always assume discrete values only, whose minimum distance is limited by the resolution of the measuring instrument or the measurement procedure.*

In terms of Statistical Tolerancing the interaction of various random influences can be illustrated by the example of processing a shaft on a turning lathe, which is subject to a lot of factors influencing the process result (diameter of shaft):

- Deformation due to insufficient rigidity of machine
- Distortion of workpiece
- Elastic deformation of workpiece and cutting tool
- Wear of bearing
- Inherent vibrations (resonances) of machine
- Variations in speed and feed rate, etc.

Overlaying the influences produces a diameter which according to the Central Limit Theorem of Statistics is approximately normally distributed.

2.4. Distribution of Characteristics and Tolerance

Actual values of characteristics must be within the respective tolerance. However, there is no stipulation for the distribution of the characteristic values within the tolerance. Based on the assumption of the most unfavorable case, that all values of the characteristic lie at the tolerance limit, the need for arithmetic tolerancing is derived.

Instead of this, in the scope of the calculation procedures of Statistical Tolerancing it is assumed that the values of a characteristic are subject to a certain distribution within the tolerance boundaries. A distribution is characterized by some parameters like mean and standard deviation.

> *REMARK:*
>
> *Stating solely the mean value and the standard deviation cannot give an indication of a specific distribution, such as normal distribution.*

Therefore, it is necessary to consider in detail the relationship between the tolerance of a characteristic and the standard deviation of specific distributions.

We confine ourselves, in the following sections, to discussions on rectangular, triangular and normal distribution. Apart from them, the literature also deals for instance with a trapezoidal distribution having different slopes of flanks (cp. [2], p. 153).

2.4.1. Variance of a Rectangular Distribution

We consider in the following the actual value of a part characteristic as random variable *X*. The probability, that the variable *X* assumes a value *x* in the range between *x* and *x* + d*x* is equal to the product *f*(*x*)d*x*, where *f*(*x*) is the so-called probability density.

The probability density for a characteristic, whose values are distributed uniformly over the entire tolerance $T = (G_o - G_u)$ bounded by the boundaries G_u and G_o has the value $f(x) = 1/T$ within this range and is equal to zero for all *x* outside *T*. For the mean μ is valid:

$$\mu = \frac{G_u + G_o}{2} . \qquad (2.8)$$

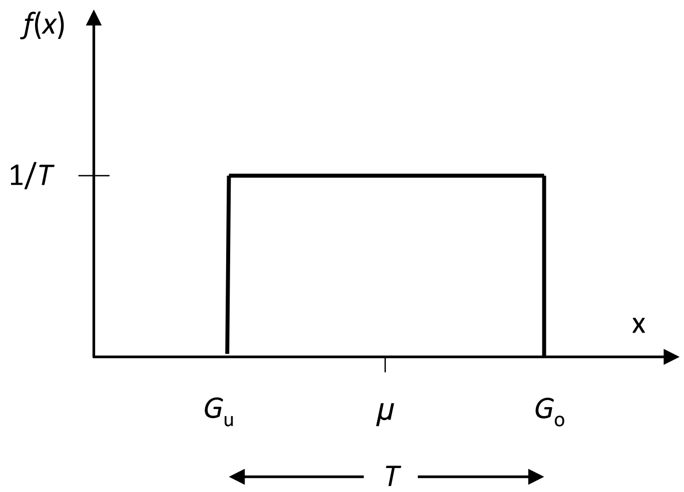


Fig. 2.7: Probability density *f*(*x*) of a rectangular distribution

The graphical representation of this function explains the term rectangular distribution. The area of the rectangle corresponds to the probability that *X* assumes any value *x* between G_u and G_o. It has the value one. The variance σ^2 of a continuous distribution is defined by

$$\sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\, dx . \qquad (2.9)$$

As in the case considered here, *f*(*x*) is equal to zero outside the interval $[G_u, G_o]$, it is enough to carry out the integration over this interval for calculating the variance of the rectangular distribution:

$$\sigma^2 = \int_{G_u}^{G_o} \left(x - \frac{G_u + G_o}{2} \right)^2 \frac{1}{T}\, dx = \frac{T^2}{12} . \qquad (2.10)$$

Rectangular distribution is an often-used conservative method, because the probability of occurrence at the tolerance limits is just as high as in the middle of the tolerance. It should therefore be applied as the input distribution whenever the actual tolerance distribution remains unclear. It is also typical for manufacturing processes involving systematic influencing factors such as tool wear or changes to process parameters. This applies, in particular, to all tool-related characteristics such as in injection-molding and punching. In this, rectangular distribution can be seen as the envelope of a varying normal distribution (varying due to tool wear for example).

2.4.2. Variance of a Triangular Distribution

The probability density of the triangular distribution in the interval [G_u , G_o] is given by:

$$f(x)=\begin{cases} \frac{4}{T^2}(x-G_u)\text{for } G_u < x < \mu \\ -\frac{4}{T^2}(x-G_o)\text{for } \mu \le x < G_o \end{cases} \tag{2.11}$$

The area of the triangle corresponds to the probability that *X* assumes any value *x* between G_u and G_o. Its value is equal to one. As the base of this triangle corresponds to the width of the rectangle (cp. Rectangular distribution), *f*(*x*) must have the value 2/T at the position $\mu = (G_u + G_o)/2$.

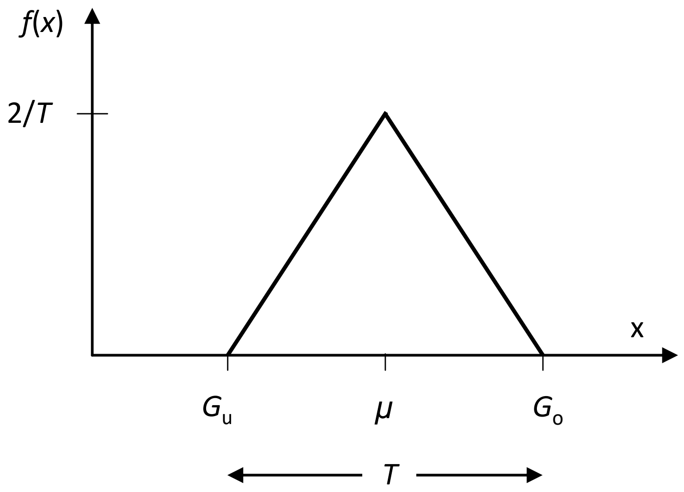


Fig. 2.8: Probability density of a triangular distribution

Because of the symmetry of the function, for calculating the variance of the triangular distribution it is enough to determine the integral in the limits [G_u , μ] and to multiply by two:

$$\sigma^2 = 2\int_{G_u}^{\mu}(x-\mu)^2\frac{4}{T^2}(x-G_u)\,dx = \frac{T^2}{24}\,. \tag{2.12}$$

Triangle distribution is typical for small-lot production runs, for tool corrections up to the center characteristic, and as an approximation to the normal distribution.

2.4.3. Variance of a Normal Distribution

Due to its characteristic shape, the graphical representation of the probability density of normal distribution is also known as a Gaussian bell curve. It is described by the mathematical equation:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \tag{2.13}$$

The probability density is unambiguously defined by the parameters μ and σ. In this, μ is the mean value of the distribution and σ is its standard deviation.

> *REMARK:*
>
> *Stating the mean value and standard deviation cannot be applied as a simple method of indicating a normal distribution.*

The probability density of normal distribution is illustrated in the following diagram.

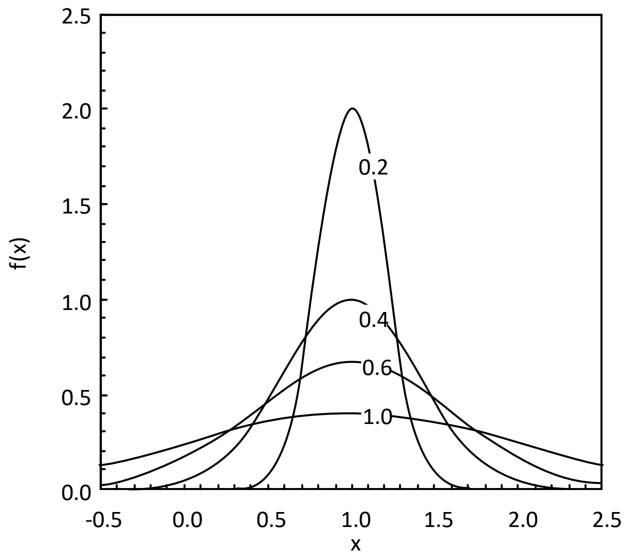


Fig. 9: Graphical representation of the probability density of normal distribution dependent on the scattering parameter σ where μ = 1.

The normal distribution has a number of special features which can be seen from its functional equation and graphical representation.

- The curve is symmetrical to the mean μ.
- At $\mu - \sigma$ and $\mu + \sigma$ the curve has inflection points.
- The curve runs from $x = -\infty$ to $x = +\infty$.

A tolerance is frequently given as the mean ± multiple of the standard deviation. This defines what percentage of the values is expected to be within the tolerance as well as the fraction nonconforming outside the tolerance limits. The percentage of values in the tolerance is termed the confidence number γ.

For different confidence numbers γ of a standard normally distributed random variable the following table summarizes the expansion factors *k* and corresponding C_P values.

Expansion factor *k*	Interval	C_p value	Confidence number γ
2	$-1\sigma < z \leq 1\sigma$	0.33	≈ 68.2689 %
4	$-2\sigma < z \leq 2\sigma$	0.66	≈ 95.4500 %
6	$-3\sigma < z \leq 3\sigma$	1	≈ 99.7300 %
8	$-4\sigma < z \leq 4\sigma$	1.33	≈ 99.9937 %
10	$-5\sigma < z \leq 5\sigma$	1.66	≈ 99.99994 %

Table 2.1: Confidence number for a standard normally distributed random variable

The relation between variance and tolerance can be depicted as

$$\sigma^2 = \frac{T^2}{k^2}. \tag{2.14}$$

Applying normal distribution is typical for tolerances of dimensions obtained in metal-cutting manufacturing. However, it does not take account of systematic effects such as tool wear. Normal distribution is thus a very ideal distribution.

2.5. Process Capability Indices

Process capability C_p is a measure of the performance which the process could deliver if optimally set up. In this method, the actual process spread is compared against the specified tolerance. For normal distributed values it can be calculated as

$$C_p = \frac{T}{6\sigma}\,. \qquad (2.15)$$

If the process capability C_p cannot be maintained for an assembly characteristic, usually the tolerance of the individual characteristics must be reduced.

The process capability index C_{pk} is a measure for the variation of a characteristic within the tolerance, considering the average position of the process. For normal distributed values applicable:

$$C_{pk} = \frac{D}{3\sigma}\,. \qquad (2.16)$$

D is the minimum distance of the characteristic distribution mean to one of the limits G_o or G_u respectively.

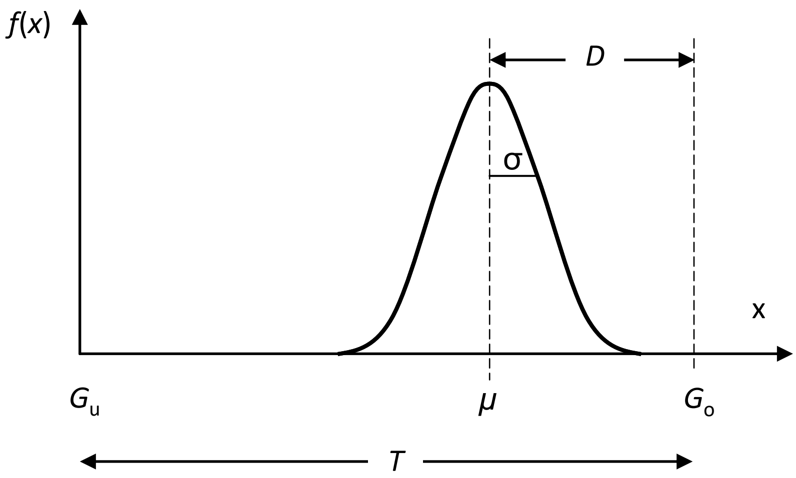


Fig. 2.10: Representation to illustrate C_{pk} (schematically)

If the process capability index C_{pk} cannot be maintained for an assembly characteristic, the mean values of the individual characteristics must usually be adjusted.

Assuming that the process is "centered", i.e. the process average coincides with the midpoint of the tolerance (that is to say: $D = T/2$), then C_{pk} is equal to C_p.

The C_p value of an assembly characteristic is, of course, not a process capability index in the original sense. However, it can be used as a formal variable, for example, to assess the functional reliability of the assemblies. Some details to this subject could be found in [6].

3. Arithmetic Tolerancing

3.1. Application to Linear Characteristic Chains

Analogous to the linking of individual characteristics to form an assembly characteristic, individual tolerances are linked to form a tolerance of the assembly characteristic. It is often determined on the basis of theoretical considerations or on the basis of experience, taking into account their significance for:

- Function,
- Reliability,
- Lifetime,
- Optical quality (appearance),
- Compliance with legal and official regulations.

As result of a design calculation, tolerances for individual characteristics are then determined (tolerance synthesis). To the contrary, if individual tolerances are given, then the assembly tolerance can be calculated by inversion of this concept (tolerance analysis).

The actual values of characteristics must be within the tolerance limits. If the most unfavorable case is assumed, that all actual values of individual characteristics have the maximum permissible deviation from the nominal value, i.e., are lying at the tolerance limit, all individual tolerances T_i must be added arithmetically to obtain the assembly tolerance, independent of the respective contributory direction of the individual characteristics (cp. Fig. 2.1). This is valid only if nominal value and midpoint of the tolerance are the same. Otherwise, the differences of the lower and upper limit to the nominal value are to be counted in accordance with the contributory direction. The tolerance of the assembly characteristic is then also asymmetric to its nominal value.

Thus:

$$T_a = T_1 + T_2 + \ldots + T_k = \sum_{i=1}^{k} T_i \ . \qquad (3.1)$$

This instance is called “Arithmetic Tolerancing”. Other names for this procedure are: “Maximum-Minimum Method” or “Method of Absolute Interchangeability”.

REMARKS:

The term “absolute interchangeability” of a part denotes its property that, without any limitation to the aforementioned design criteria (function, reliability, ...) of the assembly, the part must be able to be replaced by any other part from the same production batch.

Arithmetic Tolerancing is frequently also termed "Worst-Case Tolerancing", because it is assumed that all characteristics occur exactly at their upper or lower tolerance limits. However, this only represents a conservative method in the linear case. If, for example, tilting influences are to be taken into account, then to determine the worst case a nonlinear characteristic chain must be used.

EXAMPLE:

Given are five individual parts with characteristics and tolerances. Required are the assembly characteristic and its tolerance after assembly.

It is to be noted that in columns 5 and 6 the algebraic sign and the index change when the nominal characteristic is negative (according to the contributory direction). For the nominal characteristic -23.8 A_U *= -0.02 becomes* A_O' *= +0.02.*

As the assembly characteristic we find $0.1^{+0,05}_{-0,07}$ *and as the center of assembly characteristic* 0.09 ± 0.06*, and also* T_a *= 0.12.*

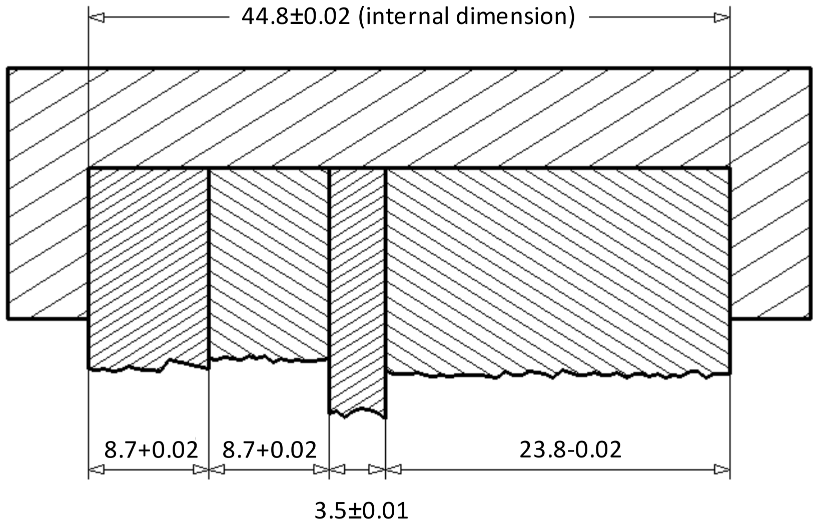


Fig. 3.1: Assembly group with nominal characteristic values and tolerances

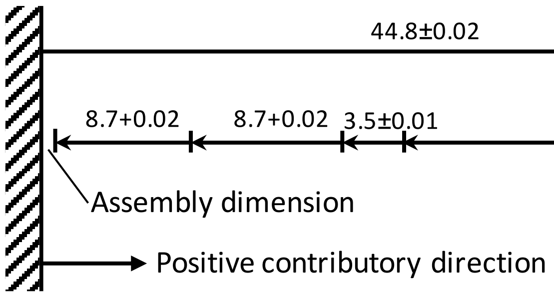


Fig. 3.2: Schematic representation of the nominal values considering the contributory direction

1	2	3	4	5	6	7	8
	Arithmetic addition of the tolerances						
Di-men-sion	Nominal value	Allowances according to drawing		Allowances according to contrib. direction		Tolerance	Center of assembly character.
	N	A_u	A_o	A_u'	A_o'	T_i	C
A	+44.8	-0.02	+0.02	-0.02	+0.02	0.04	+44.8
B	-23.8	-0.02			+0.02	0.02	-23.79
C	-3.5	-0.01	+0.01	-0.01	+0.01	0.02	-3.5
D	-8.7		+0.02	-0.02		0.02	-8.71
E	-8.7		+0.02	-0.02		0.02	-8.71
$\sum +$	+44.8						
$\sum -$	-44.7						
$\sum$	+0.1			-0.07	+0.05	0.12	+0.09
	s			A_{su}	A_{so}	T_a	C_s

Table 3.1: Calculation sheet for calculation of the arithmetical addition of the assembly tolerance.

3.2. Application to Nonlinear Characteristic Chains

Linear characteristic chains permit clearly defined calculation of tolerance overlays. As already shown in section 2.1, however, characteristics – and thus also tolerances – are also overlaid in nonlinear mode. As an example of a nonlinear dimension chain, the distance between two holes was discussed whose positions on two axes perpendicular to each other are given by their respective distance from a reference point:

$$z = \sqrt{x^2 + y^2}\ . \tag{3.2}$$

The dimension chain is no longer linear, so the tolerance calculation derived in the previous section cannot be applied directly.

In the case of nonlinear overlays of characteristics, more complex dependencies of individual characteristics may result which can then only be depicted analytically with great effort. Nonlinear characteristic chains also often occur in electrical engineering or physical tasks; examples include voltage dividers and transistor circuits.

The only available method for arithmetical tolerance calculation of nonlinear characteristic chains is linearization of the characteristic chain, introduced in section 2.2 as the Law of Error Propagation. Since this linearization is based on the Taylor's series expansion of the function, it is only useful where:

- the nonlinearity at the operating point is not excessive, and
- the tolerance to be calculated is small.

If one of these preconditions is not met, a statistical tolerance simulation must be carried out.

To derive the linearization method, it is first assumed that the assembly characteristic to be toleranced is dependent only on one individual characteristic. The function $z(x)$ can be expanded by the reference point x_0 as a Taylor's series. It is produced as:

$$z(x_0 + \Delta x) = z_0 + \Delta z = z(x_0) + \frac{1}{1!}\left.\frac{\partial z}{\partial x}\right|_{x_0} \Delta x + \frac{1}{2!}\left.\frac{\partial^2 z}{\partial x^2}\right|_{x_0} \Delta x^2 + \dots\ . \tag{3.3}$$

If the deviation Δx is small, as can be assumed in tolerance calculations, the Taylor's series can be aborted after the linear element. For the tolerance Δz this produces the linear approximation:

$$\Delta z \approx \left.\frac{\partial z}{\partial x}\right|_{x_0} \Delta x = E_x \Delta x\ . \tag{3.4}$$

The magnitude E_x with which a change Δx in the individual characteristic x impacts on the characteristic z is termed the sensitivity. The Taylor's series is used to approximate the nonlinear task in linear mode. These correlations are represented graphically in the following diagram.

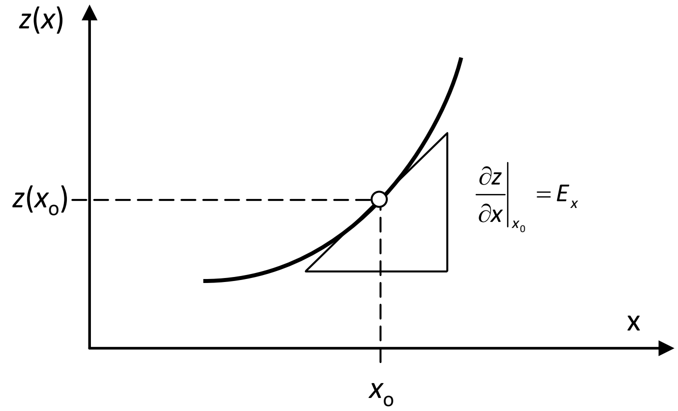


Fig. 3.3: Linearization at the operating point

This linearization method can be generalized to functions of multiple individual characteristics x_i. The functional relation between the characteristics x_i and the assembly characteristic z which has to be tolerated has the form:

$$z = f(x_1, x_2, \dots, x_n). \tag{3.5}$$

The function can be described as a multidimensional Taylor's series. If the Taylor's series is aborted after the linear element, the result is the total differential:

$$\Delta z \approx E_{x1}\Delta x_1 + E_{x2}\Delta x_2 + \dots + E_{xn}\Delta x_n = \Delta z_1 + \Delta z_2 + \dots + \Delta z_n. \tag{3.6}$$

The Taylor's series linearizes the problem. The nonlinear relation produces a linear characteristic chain with the summands $\Delta z_i = E_{xi}\Delta x_i$.

Following this transformation, the tolerance calculation methods set out in the previous section can be applied. The arithmetical tolerance calculation based on this equation is then:

$$T = \sum_{i=1}^{n} \left|E_{xi}T_i\right| = \left|E_{x1}T_1\right| + \left|E_{x2}T_2\right| + \dots + \left|E_{xn}T_n\right|. \tag{3.7}$$

EXAMPLE:

The distance between two holes is calculated according to equation 3.2. The tolerance of the distance Δz can be estimated with the aid of the complete error differential by way of the individual tolerances Δx and Δy as:

$$\Delta z \approx \left.\frac{\partial z}{\partial x}\right|_{x_0, y_0} \Delta x + \left.\frac{\partial z}{\partial y}\right|_{x_0, y_0} \Delta y = E_x \Delta x + E_y \Delta y$$

The equation produces the sensitivity E_x, with which a tolerance Δx impacts on the assembly tolerance Δz

$$E_x = \frac{1}{2}\frac{2x_0}{\sqrt{x_0^2 + y_0^2}} = \frac{x_0}{\sqrt{x_0^2 + y_0^2}},$$

as well as the sensitivity E_Y with which a tolerance Δy impacts the assembly tolerance Δz

$$E_y = \frac{1}{2}\frac{2y_0}{\sqrt{x_0^2 + y_0^2}} = \frac{y_0}{\sqrt{x_0^2 + y_0^2}}.$$

The dependencies are illustrated on basis of the following numerical example.

Dimension	Nominal charact.	Tolerance limits		Tolerance	Center charact.	Sensitivities	Tolerance influence
		Min.	Max.	T_i		E_i	$E_i T_i$
x	40	- 0.1	0.1	0.2	40	0.8	0.16
y	30	- 0.1	0.1	0.2	30	0.6	0.12
z	50				50		0.28

The distance z for the specified nominal characteristics and tolerances is thus:

z = 50 mm ± 0.14 mm.

4. Statistical Tolerancing

In arithmetic tolerancing minimum and maximum characteristics are overlaid such that the effect has maximum influence on the characteristic chain. More complex products often feature lots of individual components, and thus many different characteristics. As the number of components increases, the characteristic chain to calculate the assembly tolerance becomes ever longer. In arithmetic tolerancing, therefore, the assembly tolerance increases steadily. But because the probability decreases that all components will simultaneously have an actual value at the tolerance limits, the tolerance calculation should take into account the statistical distribution of the individual tolerances. This approach results in statistical tolerancing.

Statistical tolerancing is based on the idea that random positive and negative deviations of individual characteristics from their respective nominal values balance themselves out. The process of tolerance overlaying is interpreted as a random process in which each characteristic x_i is modeled as a random variable with a probability density $f(x_i)$. Thus, the assembly characteristic z is likewise a random variable with a resultant probability density $f(z)$. If the probability density of the assembly characteristic z is known, the tolerance can be determined by way of a confidence number γ.

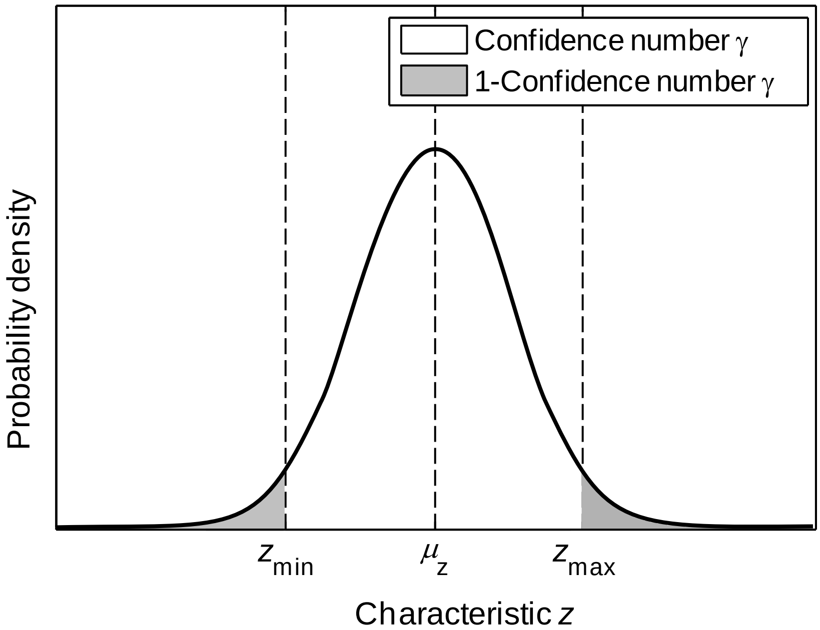


Fig. 4.1: Relation between tolerance zone and probability distribution

A statistical tolerance calculation can be carried out by different methods. They differ in their preconditions and in the effort required to implement them. The following representation assumes statistically independent individual characteristics and depicts a linear and nonlinear tolerance calculation for that assumption. The way to handle dependent individual characteristics is outlined in the appendix.

4.1. Application to Linear Uncorrelated Characteristic Chains

4.1.1. Statistical Tolerance Calculation by Convolution

In statistical tolerancing an individual characteristic x_i is described as a random variable. The tolerance overlay produces a sum of random variables which is itself in turn a random variable.

$$z = x_1 + x_2 + ... + x_n. \quad (4.1)$$

The probability density of the sum of random variables is produced from convolution of the individual probability densities:

$$f(z) = f(x_1)^* f(x_2)^* \ldots ^* f(x_n). \tag{4.2}$$

If the probability densities of the individual characteristics are known, the probability density of the assembly characteristic can be determined by convolution of the individual probability densities. The tolerance of the assembly characteristic can be determined from this distribution with an indication of certainty.

Calculation of the statistical distribution by means of convolution is always mathematically correct, assuming uncorrelated individual characteristics and linear characteristic chains. The effort involved in numerical calculation and evaluation of the distribution is high, however, and can only be handled by appropriate software tools such as the MATLAB program. For this reason, statistical tolerancing in practice is estimated with approximation solutions.

4.1.2. Statistical Tolerance Calculation in Case of Normally Distributed Characteristics

If the individual characteristics x_i are normally distributed with the mean value μ_i and the standard deviation σ_i, the assembly characteristic z as the sum of all individual characteristics likewise has a normal distribution. It has the mean value:

$$\mu = \mu_1 + \mu_2 + \ldots + \mu_n. \tag{4.3}$$

The variances of the individual characteristics are added together to form the variance of the assembly characteristic:

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2. \tag{4.4}$$

Typically, a tolerance $T_i = \pm 3\sigma$ with a confidence number $\gamma = 99.73$ % is assumed for the individual characteristics, although a confidence number of 99.9937 % is specified. This implicitly takes into account a possibly non-central tolerance position. In this case the relation between the standard deviation σ_i and the tolerance T_i is:

$$\sigma_i = \frac{T_i}{6}. \tag{4.5}$$

Subject to this assumption of identical expansion factors k for the individual characteristics x_i and the assembly characteristic z, the tolerance calculation is simplified as:

$$T^2 = T_1^2 + T_2^2 + \ldots + T_n^2. \tag{4.6}$$

EXAMPLE:

The following table displays the specification for five characteristics of a linear tolerance chain. The nominal characteristic and the minimum and maximum deviation according to the drawing are specified. It is assumed that all lengths have a normal distribution and can be regarded as uncorrelated. The tolerance specified by the manufacturer is interpreted as $\pm 3\sigma$, so the standard deviation in each case is produced from one sixth of the specified tolerance.

In this, the tolerance in statistical tolerancing is calculated from:

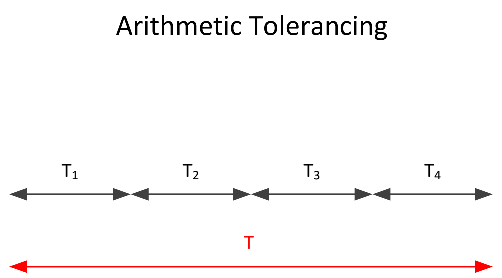
$$T = \sqrt{T_1^2 + T_2^2 + \ldots + T_5^2} = 0.0566 \text{ mm}. \tag{4.7}$$

As expected, the statistical calculation exhibits smaller tolerances than the arithmetic tolerance calculation. This advantage rises as the number of components involved increases.

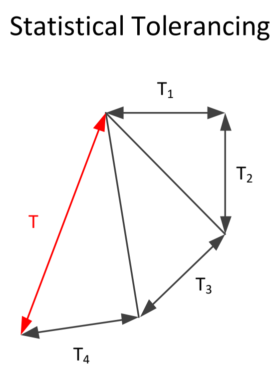
Charact-eristic	Nom-inal charac teristic	Contrib-utory direction	Tolerance according to drawing		Tolerance in contributory direction		Center characteris tic	Tolerance	
			Min.	Max.	Min.	Max.		Arith-metic	Statistical
1	44.8	1	- 0.02	+ 0.02	- 0.02	+ 0.02	44.8	0.04	0.04
2	23.8	1	- 0.02	0	- 0.02	0	23.79	0.02	0.02
3	3.5	1	- 0.01	+ 0.01	- 0.01	+ 0.01	3.50	0.02	0.02
4	8.7	1	0	+ 0.02	0	+ 0.02	8.71	0.02	0.02
5	8.7	1	0	+ 0.02	0	+ 0.02	8.71	0.02	0.02
Total	89.5				- 0.05	+ 0.07	89.51	0.12	0.0566

Table 4.1: Tolerance calculation for a linear tolerance chain with five characteristics

In statistical tolerancing, the tolerance of the assembly characteristic is much smaller than in arithmetic tolerancing. The difference in tolerance calculation can be illustrated graphically, assuming normally distributed individual characteristics. Whereas in arithmetic tolerancing the tolerances are added together in linear mode, in statistical tolerancing they are added quadratically. The addition can be traced back to Pythagoras's Theorem, and can be depicted by right triangles. Owing to the quadratic addition, the assembly tolerance is much smaller especially where there are lots of individual tolerances.



$$T = |T_1| + |T_2| + \ldots + |T_n|$$



$$T^2 = T_1^2 + T_2^2 + \ldots + T_n^2$$

Fig. 4.2: Visualization of arithmetic and statistical tolerancing

The example of linear overlaying shows how easily statistical tolerancing can be carried out if the individual characteristics are normally distributed. This precondition is not always met, however, so the model has to be extended.

4.1.3. Statistical Tolerancing in Case of Characteristics with Known Distribution

The assumption of normally distributed individual characteristics is in many cases justified at least to a good degree of approximation. Some individual characteristic distributions differ significantly from a normal distribution, however. Typical distributions are summarized in the table.

Distribution	Standard deviation σ	Application case
Rectangular distribution	$\sigma = \frac{T}{\sqrt{12}}$	Tool-related characteristics (e.g. injection-molding, punching)
Symmetrical triangular distribution	$\sigma = \frac{T}{\sqrt{24}}$	Tool corrections
Normal distribution	$\sigma = \frac{T}{k}$	Dimensional tolerances with no systematic influence (for selection of expansion factor *k* see section 2.4.3).

Table 4.2: Distributions of characteristics and their typical applications

Rectangular distributions occur in relation to wear of tools for example. The characteristic of the produced part changes continually until the tool being used is replaced. Then the wear process starts again, so that the characteristic of the produced parts is equally distributed within defined limits. Uneven distributions occur whenever the measured value has a natural limit.

If the individual characteristics are not normally distributed, the assembly characteristic likewise does not exhibit normal distribution. According to the Central Limit Theorem of Statistics, however, the sum of symmetrically distributed random variables is asymptotically normally distributed. Consequently, from the distributions of the individual characteristics x_i the mean values μ_i and the standard deviations σ_i are determined. Following the Central Limit Theorem of Statistics and the Error Propagation Law, overlaying the individual probability densities results in an asymptotic normal distribution with the variance:

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2 \,. \qquad (4.8)$$

The following preconditions must be met for statistical tolerancing by means of the Error Propagation Law:

- Individual characteristics are uncorrelated random variables. The calculation of the statistical tolerance is based on the fact that the variables are not mutually correlated. The special case of correlated characteristics is discussed in the appendix.
- Characteristic chain is linear. The characteristic chain must be known and be capable of representation by a linear equation. In the case of nonlinear characteristic chains, the characteristic chain is linearized. Alternatively, a Monte Carlo simulation can be performed.
- Characteristic chain has at least four variables. The Central Limit Theorem of Statistics only delivers sufficiently precise results when at least four quantities are set against each other. Otherwise, the assumption of normal distribution for the assembly characteristic *z* is not justified.
- Individual characteristics are of the same magnitude. The statistical compensation of negative and positive deviations of actual values from nominal values can function only when the random variation ranges of the actual values and therewith also their tolerances are approximately equal.
- Probability densities $f(x_i)$ of the individual characteristics x_i are known. Statistical tolerancing is based on the applied probability densities. Errors in the underlying probability density or changes in the distribution over time result in incorrect tolerance calculations.

- Probability densities $f(x_i)$ of the individual characteristics x_i are symmetrical. Statistical compensation of negative and positive deviations of a characteristic is only mean-free if positive and negative deviations are equally probable.

EXAMPLE:

Linear characteristic chains occur not only in relation to geometric characteristics. The diagram depicts a series configuration of resistors as an example from electrical engineering. The individual resistances R_1 to R_3 are added together to form the total resistance R:

$$R = R_1 + R_2 + R_3 \,. \tag{4.9}$$

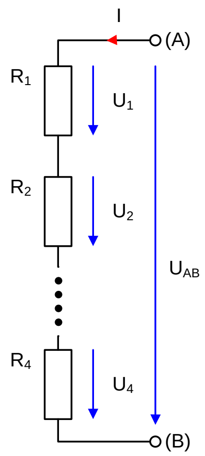


Fig. 4.3: Series configuration of resistors

The first resistor is normally distributed with a $\pm 3\sigma$ value of 20 Ω; the second has an equal distribution in the two tolerance limits of 95 and 105 Ω; and the third has a triangular distribution in the tolerance limits of 92 and 108 Ω. This results in a standard deviation σ_1 for the first resistor of:

$$\sigma_1 = \frac{T_1}{6} = \frac{20\Omega}{6} = 3.33\Omega \,. \tag{4.10}$$

The second resistor has a rectangular distribution, whereby the standard deviation σ_2 is produced from:

$$\sigma_2 = \frac{T_2}{\sqrt{12}} = \frac{10\Omega}{\sqrt{12}} = 2.89\Omega \,. \tag{4.11}$$

From the symmetrical triangular distribution of the third resistor the standard deviation σ_3 follows as:

$$\sigma_3 = \frac{T_3}{\sqrt{24}} = \frac{16\Omega}{\sqrt{24}} = 3.26\Omega \,. \tag{4.12}$$

This results in a $\pm 3\sigma$ tolerance of the total resistor of

$$T_G = 6\sigma = 6\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = 32.9\Omega \,. \tag{4.13}$$

The following diagram compares the probability density of the total resistor for a calculation by way of the Error Propagation Law and for exact calculation by convolution of the probability densities.

The probability density calculated by convolution differs slightly from the probability density calculated using the Error Propagation Law. The differences between the two methods relate to the smaller number of individual characteristics. This means that the assumed normal distribution is not exactly fulfilled. The result calculated by convolution is therefore more precise than that calculated by way of the standard deviations.

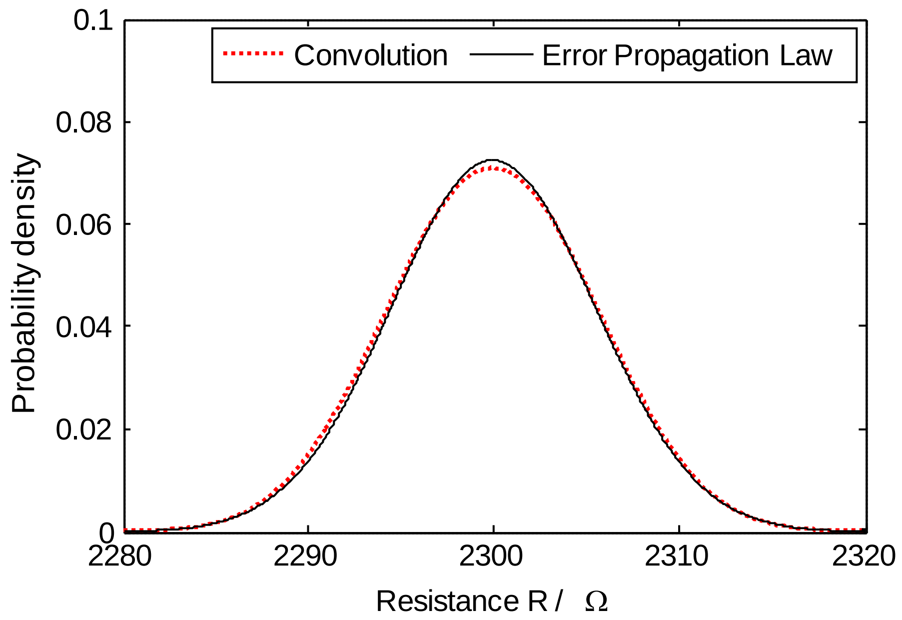


Fig. 4.4: Comparison of the probability density for a calculation by Error Propagation Law and by convolution of the probability densities

4.1.4. Statistical Tolerance Simulations

An analytical tolerance calculation quickly becomes lengthy and complex in relation to long or nonlinear characteristic chains. So, a statistical tolerance simulation is carried out instead. The most popular method of statistical simulation is Monte Carlo simulation.

In Monte Carlo simulation the assembly characteristic is described by way of a mathematical model and initially calculated for a set of individual characteristics. This calculation is repeated in the event of any variation in the individual characteristics taking into account the respective distributions of the individual characteristics when generating them. The simulation can thus be interpreted as a random experiment in which the assembly characteristic is a random variable. Each simulation result delivers a value for the assembly characteristic.

The simulation process is depicted in the following figure as a flowchart.

In the first step random numbers are formed for all individual characteristics taking into account their probability densities. These input quantities are used to calculate the assembly characteristic. From the calculated values, predicted intervals for future values of the assembly characteristic can be calculated.

The simulation process is implemented until the certainty is sufficiently high. The statistical distribution of the assembly characteristics is derived from the various simulation results.

Monte Carlo simulation is thus a purely numerical tool which determines the assembly tolerance by targeted trials. The individual characteristics may have different distributions which must be taken into account when generating the random numbers. The distributions should not be symmetrical, the individual tolerances may also strongly differ from each other.

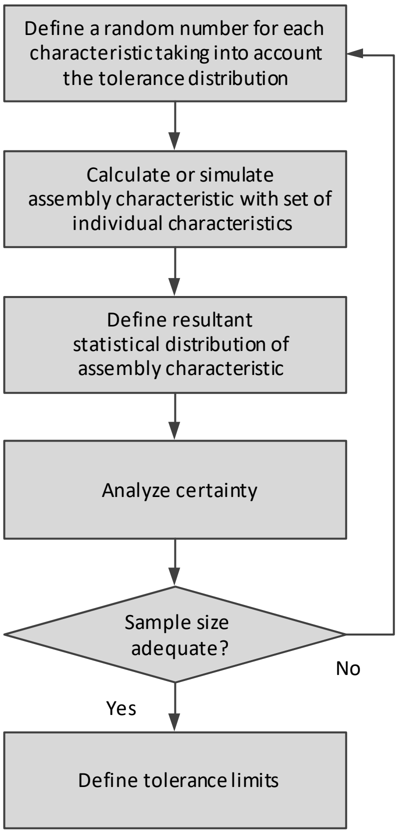


Fig. 4.5: Flowchart depicting a Monte Carlo simulation

EXAMPLE:

To illustrate Monte Carlo simulation, the resistance chain is simulated with individual characteristic distributions such that the first resistance is normally distributed with a $\pm 3\sigma$ value, the second has a rectangular distribution in the two tolerance limits and the third has a triangular distribution in the two tolerance limits.

To generate the random numbers, the resistance R_1 is applied as normal distribution with a mean value of μ_1 = 100 Ω and a standard deviation σ_1 of:

$$\sigma_1 = \frac{T_1}{6} = \frac{20\Omega}{6} = 3.33\Omega \qquad (4.14)$$

The second resistance has a rectangular distribution between 95 and 105Ω. For the third resistance random numbers with a triangular distribution between 92 and 108Ω are generated. Then the sum of each group of random numbers is calculated and the result is statistically evaluated.

The following diagram presents the result of a Monte Carlo simulation for a series configuration of resistors.

In a first approximation, the result shows a normal distribution with a mean value of 300.15Ω and a standard deviation s = 5.249Ω. The predicted interval for future resistance values can be calculated from the data set assuming a normal distribution. The predicted interval for future observations in the case of a normally distributed population with an unknown mean μ and unknown variance σ_2 is calculated as:

$$PR\left\{\bar{R} + c_1 s\sqrt{1+\frac{1}{n}} \le R_{n+1} \le \bar{R} + c_2 s\sqrt{1+\frac{1}{n}}\right\}. \qquad (4.15)$$

where the constants c_1 and c_2 are produced from the inverse t-distribution with n - 1 degrees of freedom in the case of a confidence number γ.

A confidence number γ = 99.73 % with a sample set of n = 1000 results in a tolerance of:

$$T_{U,MC1} = c_2 s\sqrt{1+\frac{1}{n}} - c_1 s\sqrt{1+\frac{1}{n}} = 31.592\Omega\ . \tag{4.16}$$

The value largely corresponds to the value of the statistical tolerance calculation using the Error Propagation Law.

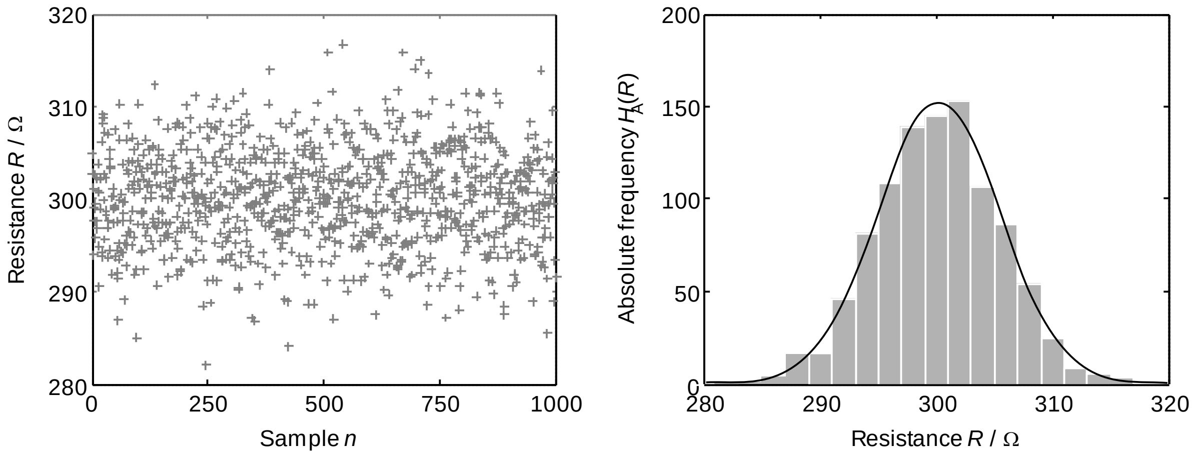


Fig. 4.6: Definition of total resistance by Monte Carlo simulation

The Monte Carlo simulation is implemented until the certainty is sufficiently high. To keep the simulation time short, it is necessary to estimate the required sample size. In doing so, a distinction must be made as to whether the output quantity can be regarded as normally distributed or not.

4.1.4.1. Estimation of Sample Size in Case of Normal Distribution of Assembly Characteristic

In the case of normally distributed individual characteristics, the required sample size can be derived from the ratio of the standard deviation to the confidence interval of the standard deviation. As the sample size increases, the standard deviation confidence interval becomes smaller. However, the standard deviation of the assembly characteristic is determined by the spread of the individual characteristics involved and remains largely constant. If the confidence interval of the standard deviation is a magnitude smaller than the standard deviation of the assembly characteristic, the simulation can be stopped.

EXAMPLE:

Figure 4.6 shows that the total resistance is approximately normally distributed. A hypothesis test can also be used to show that the assumption of a normal distribution at least does not have to be discarded. Thus the confidence interval of the standard deviation can be estimated by way of the Chi² distribution with n - 1 degrees of freedom. The resultant confidence interval is:

$$KONF\left\{\frac{s^2(n-1)}{c_1} \ge \sigma^2 \ge \frac{s^2(n-1)}{c_2}\right\}, \tag{4.17}$$

where the constants c_1 and c_2 are calculated from the inverse Chi² distribution (F^{-1}) with n - 1 degrees of freedom as:

$$c_1 = \mathrm{F}^{-1}\left(\frac{1-\gamma}{2}\right),\ c_2 = \mathrm{F}^{-1}\left(\frac{1+\gamma}{2}\right). \tag{4.18}$$

As the sample size increases, the confidence interval becomes smaller due to the constant c_1 and c_2. Conversely, the standard deviation of the total resistance R is determined by the spread of the components involved and remains largely constant. Figure 4.7 shows these quantities as a function of the sample size.

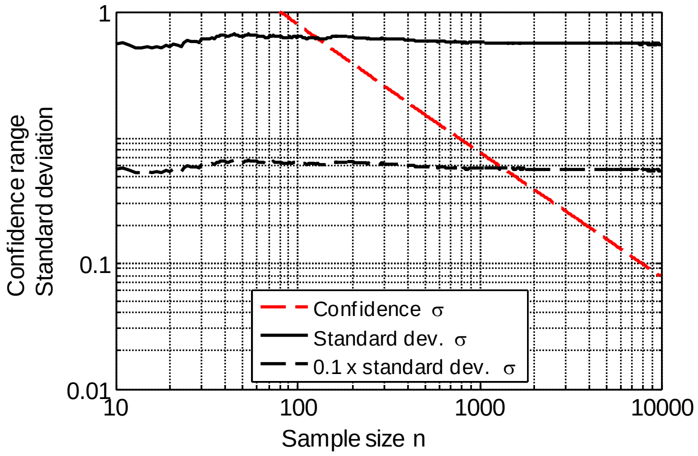


Fig. 4.7: Representation of the standard deviation and the length of the confidence interval for the total resistance as a function of the sample size *n*

The confidence interval does not have to become continually shorter. Some extreme values may initially cause the standard deviation to increase and so extend the confidence interval. As the sample size rises, however, a shortening of the confidence interval is to be expected. In this example, with a sample size of n = 1310 the confidence range of the standard deviation is less than 10 % of the standard deviation. The simulation can thus be aborted at n = 1310.

4.1.4.2. Estimation of Sample Size for Output Quantities with Any Distribution

An advantage of the Monte Carlo method is that it can also be applied to extremely nonlinear processes and to any distributions. Under these conditions, however, normally distributed assembly characteristics are not necessarily to be expected. In this case the tolerance must be determined numerically by the approximated distribution function. The confidence interval is thus produced from the lower and upper limit:

$$y_{min} = F^{-1}\left(\frac{1}{2}(1-\gamma)\right),\ \ y_{max} = F^{-1}\left(\frac{1}{2}(1+\gamma)\right), \qquad (4.19)$$

with the confidence γ and F^{-1} - the numerically determined inverse distribution function. Its resolution ΔP is derived directly from the number of simulations carried out as:

$$\Delta P = \frac{1}{n}\,. \qquad (4.20)$$

This reveals one of the disadvantages of this method – namely the high computing power required to generate the data sets. To give an indication in relation to a failure rate of 1 ppm, at least a million calculations have to be performed.

EXAMPLE:

For a voltage divider the tolerance of the output voltage is determined by the numerically defined distribution function. The resolution of the probability ΔP should be 10 % of the lower limit value, which with a certainty of $\pm 3\sigma$ results in:

$$\Delta P = \frac{1}{10}\cdot\frac{1-\gamma}{2} = \frac{1}{10}\cdot\frac{1-0.9973}{2} = 1.35E-4\,. \qquad (4.21)$$

The resultant simulation size is:

$$n = \frac{1}{\Delta P} = \frac{1}{1.35E-4} = 7407\,. \qquad (4.22)$$

4.2. Application to Nonlinear Uncorrelated Characteristic Chains

For statistical tolerance calculation of nonlinear characteristic chains, two different methods are available.

- Linearization of the characteristic chain via total differential and
- nonlinear tolerance simulation (Monte Carlo simulation).

Both methods are presented in the following and illustrated by examples.

4.2.1. Linearization of Nonlinear Characteristic Chains

A function of multiple characteristics x_i generally has the form

$$z = f(x_1, x_2, \ldots, x_n)\,. \tag{4.23}$$

The function can be expanded into a Taylor's series around the point $\mathbf{x}_0$. If the Taylor's series is aborted after the linear element, the result is the total differential:

$$\begin{aligned}\Delta z &\approx \left.\frac{\partial z}{\partial x_1}\right|_{\mathbf{x}_0} \Delta x_1 + \left.\frac{\partial z}{\partial x_2}\right|_{\mathbf{x}_0} \Delta x_2 + \ldots + \left.\frac{\partial z}{\partial x_n}\right|_{\mathbf{x}_0} \Delta x_n = E_{x1}\Delta x_1 + E_{x2}\Delta x_2 + \ldots + E_{xn}\Delta x_n \\ &= \Delta z_1 + \Delta z_2 + \ldots + \Delta z_n\end{aligned} \tag{4.24}$$

This method linearizes the problem: From the nonlinear correlation as per equation 4.23 a linear characteristic chain is derived in which the individual summands Δz_i are produced from:

$$\Delta z_i = E_{xi}\Delta x_i\,. \tag{4.25}$$

Following this transformation, the tolerance calculation methods set out in the previous section can be applied. In particular, this results in the following expression for the variance of the assembly characteristic:

$$\sigma_z^2 = \sum_{i=1}^{n} \sigma_{zi}^2 = \sum_{i=1}^{n} E_{xi}^2 \sigma_{xi}^2 = E_{x1}^2\sigma_{x1}^2 + E_{x2}^2\sigma_{x2}^2 + \ldots + E_{xn}^2\sigma_{xn}^2\,. \tag{4.26}$$

In the case of identical expansion factors, the tolerance of the assembly characteristic can be defined as:

$$T_z^2 = \sum_{i=1}^{n} E_{xi}^2 T_{xi}^2 = E_{x1}^2 T_{x1}^2 + E_{x2}^2 T_{x2}^2 + \ldots + E_{xn}^2 T_{xn}^2\,. \tag{4.27}$$

EXAMPLE:

A voltage divider with the resistances R_1 and R_2 is powered by a source voltage U_{ref}.

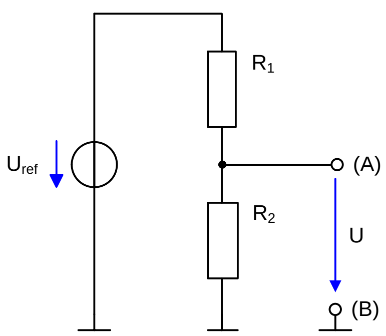


Fig. 4.8: Voltage divider as electrical engineering example of tolerance calculation

The output voltage U is calculated according to the voltage divider rule from the quantities R_1, R_2 and U_{ref} as:

$$U = \frac{R_2}{R_1 + R_2} U_{ref} \ . \tag{4.28}$$

Similar to the case of the distance between the holes, the tolerance of the output voltage can be estimated by way of the complete error differential as:

$$\Delta U \approx \frac{\partial U}{\partial R_1}\Delta R_1 + \frac{\partial U}{\partial R_2}\Delta R_2 + \frac{\partial U}{\partial U_{ref}}\Delta U_{ref} \ . \tag{4.29}$$

In this, the sensitivity with which, for example, the change in resistance ΔR_1 impacts on the change in output voltage is calculated as:

$$E_{R1} = \frac{\partial U}{\partial R_1} = \frac{-R_2 U_{ref}}{(R_1 + R_2)^2} \ . \tag{4.30}$$

The unit of sensitivity is V/Ω. Accordingly, the sensitivity converts a tolerance of the influence quantity R_1 in Ohms into a tolerance of the target quantity U in Volts. Analogously, the following is produced:

$$E_{R2} = \frac{\partial U}{\partial R_2} = \frac{R_1 U_{ref}}{(R_1 + R_2)^2} \tag{4.31}$$

and

$$E_{Uref} = \frac{\partial U}{\partial U_{ref}} = \frac{R_2}{(R_1 + R_2)} \ . \tag{4.32}$$

To calculate the assembly tolerance, the individual tolerance causes must be overlaid:

$$\Delta U = \Delta U_{R1} + \Delta U_{R2} + \Delta U_{Uref} = E_{R1}\Delta R_1 + E_{R2}\Delta R_2 + E_{Uref}\Delta R_{Uref} \ . \tag{4.33}$$

All random variables ΔU_i have the same unit; the task is thus traced back to a linear characteristic chain.

For the example, it is to be assumed that the resistances have a normal distribution with a mean value of R_0 = 100 Ω and a standard deviation of σ_R = 1 Ω. The voltage source is to be evenly distributed between 4.975 V and 5.025 V, thus having an equivalent standard deviation of σ_{Uref} = 0.0144 V. Following the Error Propagation Law, the variance of the assembly characteristic is produced from the sum of the variances of the individual characteristics. In this case the following is produced:

$$\sigma_U^2 = \sigma_{UR1}^2 + \sigma_{UR2}^2 + \sigma_{UUref}^2 = E_{R1}^2\sigma_{R1}^2 + E_{R2}^2\sigma_{R2}^2 + \ldots + E_{Uref}^2\sigma_{Uref}^2 \ . \tag{4.34}$$

This results in a standard deviation of the output voltage U of:

$$\sigma = \sqrt{E_{R1}^2\sigma_{R1}^2 + E_{R2}^2\sigma_{R2}^2 + \ldots + E_{Uref}^2\sigma_{Uref}^2} = 0.0191 V \ , \tag{4.35}$$

and tolerance $\pm 3\sigma$ of:

$$T_G = 6\sigma_U = 0.1146 V \ . \tag{4.36}$$

4.2.2. Statistical Tolerance Simulation of Nonlinear Characteristic Chains

Statistical tolerance simulation of nonlinear characteristic chains does not differ from statistical tolerance simulation of linear characteristic chains. Statistical tolerance simulation has the advantage that it precisely maps the nonlinearity and does not approximate the tolerance calculation by way of a linearization at the operating point.

EXAMPLE:

The example of the voltage divider is applied to illustrate Monte Carlo simulation. The resistances have a normal distribution with the mean value $R_0 = 100\ \Omega$ and a standard deviation of $\sigma_R = 1\ \Omega$. The voltage source is evenly distributed between 4.975 V and 5.025 V. The distributions are taken into account when generating random numbers. Then the associated output voltage is calculated for each group of random numbers and the result is statistically evaluated. The simulation can be implemented by a MATLAB program for example.

The following diagram shows the result of a Monte Carlo simulation for the output voltage of the voltage divider.

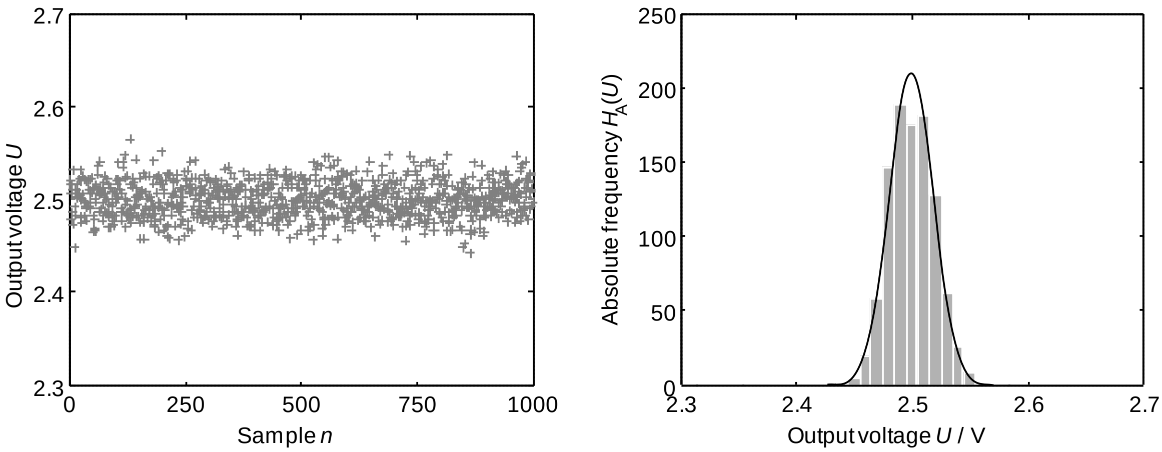


Fig. 4.9: Definition of output voltage by a Monte Carlo simulation

In a first approximation, the result shows a normal distribution with a mean value of 2.5 V and a standard deviation s = 0.0188V. The predicted interval for future output voltages can be calculated from the data set assuming a normal distribution. The predicted interval for future observations in the case of a normally distributed population with an unknown mean μ and unknown variance σ^2 is calculated as:

$$PR\left\{ \bar{x} + c_1 s\sqrt{1+\frac{1}{n}} \le x_{n+1} \le \bar{x} + c_2 s\sqrt{1+\frac{1}{n}} \right\}, \tag{4.37}$$

where the constants c_1 and c_2 are produced from the inverse t-distribution with n - 1 degrees of freedom in the case of a confidence number γ. A confidence number $\gamma = 99.73$ % with a sample set of n = 1000 results in a tolerance of:

$$T_{U,MC1} = c_2 s\sqrt{1+\frac{1}{n}} - c_1 s\sqrt{1+\frac{1}{n}} = 0.1134\,V\ . \tag{4.38}$$

The value largely corresponds to the value of the statistical tolerance calculation using the error Propagation Law.

4.3. Roadmap for Tolerance Calculation for Uncorrelated Characteristics

Sections 4.1 and 4.2 set out different methods of statistical tolerance calculation. The relation between the methods is summarized in the following diagram.

Linear characteristic chains can be toleranced arithmetically or statistically. Statistical tolerancing is particularly clear and informative in the case of normally distributed individual characteristics; the variances of the individual characteristics are simply added to the variance of the assembly characteristic. This method can also be applied where there is a sufficient number ($N \geq 4$) of individual characteristics with arbitrary symmetrical distribution. As the output is only asymptotically normally distributed, this calculation is an approximation. The exact distribution of the assembly characteristic can be calculated by way of convolution of probability densities of individual characteristics. This method is precise, but the calculation requires special programs. For statistical simulations (Monte Carlo simulation) the distribution of assembly characteristic is defined by numerical simulation.

A nonlinear characteristic chain can be linearized and then handled like a linear characteristic chain. The linearization becomes complex in the case of systems with multiple individual characteristics, however. Alternatively, a statistical simulation (Monte Carlo simulation) can be performed.

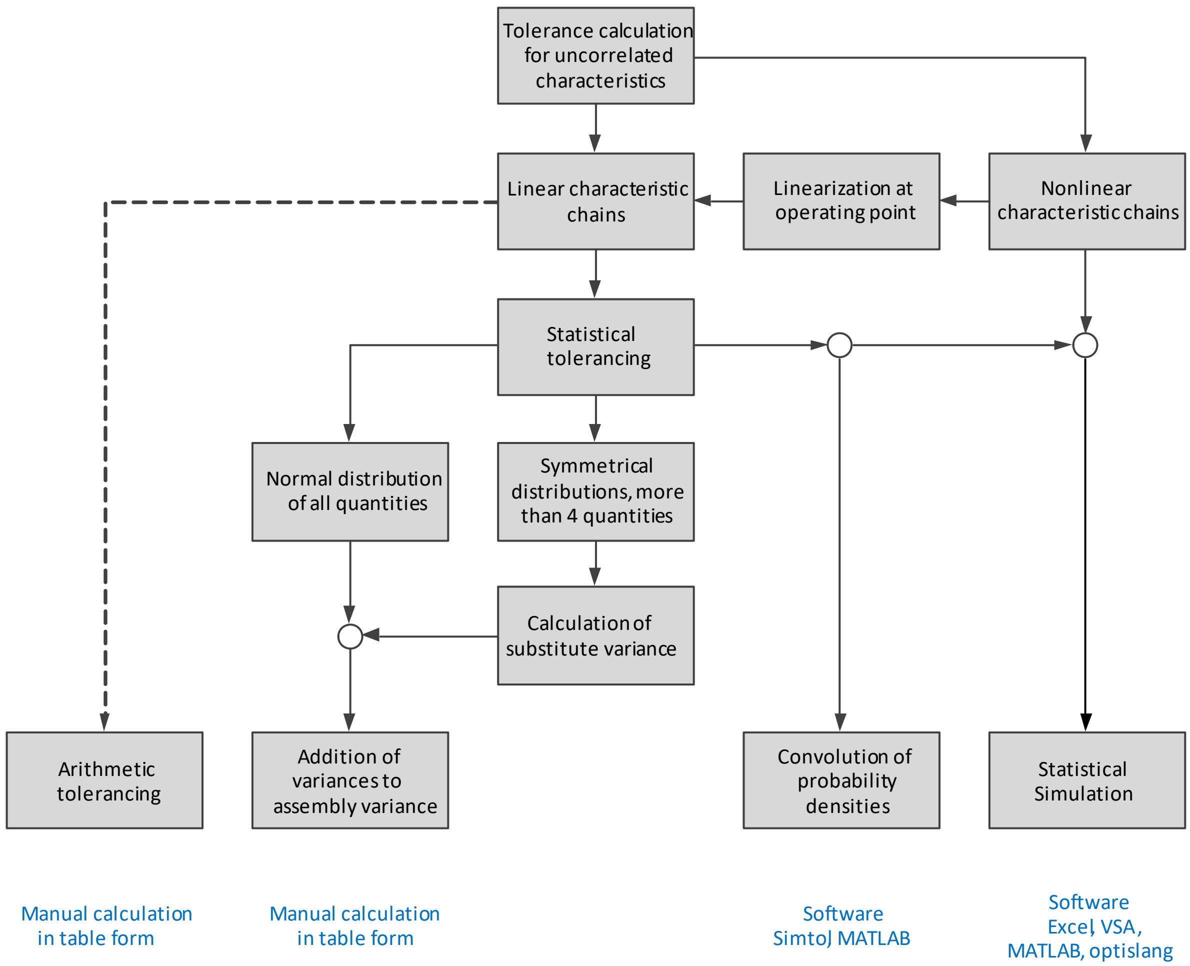


Fig. 4.10: Tolerance calculation method in case of uncorrelated characteristic chains

5. Software tools

As shown in Figure 4.10, there are various software tools for performing tolerance analyses. The following sets out the basic procedure when using such tools.

5.1. Excel

Arithmetic and linear statistical tolerance analyses can be performed by simply concatenating the worksheet cells. Relevant templates exist in practically all divisions.

Statistical tolerance simulation by the Monte Carlo method can also be performed in Excel. Each individual characteristic is then assigned a column. In that column, random numbers are determined dependent on the nominal value and characteristic distribution. On each row the random assembly characteristic is defined by way of the characteristic chain. As a result, an Excel row corresponds to one step in a Monte Carlo simulation.

The determination of distribution-dependent random numbers is described in Section 6.5

Due to the central limit theorem of statistics the assembly dimension will be approximately normally distributed under certain conditions (s. Section 4.1.3), independently of the distribution of the individual dimensions. However, if one or more of these conditions is not met, there may be deviations from the normal distribution that are not necessarily identifiable based on a histogram alone. If the simulation result is then reduced to a mere statement of mean and standard deviation, false conclusions are preprogrammed. Such false conclusions are e.g.

- unrealistic theoretical nonconforming proportions with regard to given specification limits or
- unrealistic specification limits with regard to selected theoretical nonconforming proportions

Thus, if a Monte Carlo simulation is performed using EXCEL, it is advisable to analyze the values obtained for the assembly dimension in a tool such as qs-STAT, Minitab, or Matlab with respect to the assignable best-fitting distribution model.

5.2. CAD-Assisted 3D Tolerance Analysis

For tolerance analysis of mechanical assemblies there is a number of commercial software tools which directly use the 3D design data. The preferred tool at Bosch is the software “Variation Systems Analysis (VSA)”, which runs within the “Teamcenter Visualization (TCV)” 3D viewer. This tool works internally with Monte Carlo simulation.

In contrast to the previously cited tolerance analysis methods, in CAD-assisted tolerance analysis the characteristic chain is not predefined; instead, it is determined by the software based on logic links from geometric features. Construction of the Simulation model involves four steps:

1. Definition of the tolerance features: For each component the geometric features influencing the characteristic chain are defined. They include surface areas (planes, cylinders, free-form areas, ...), edges, axes and points which are responsible for alignment of adjacent components, or limit the assembly characteristic, or serve as references for position tolerances.
2. Tolerance definition: Each tolerance feature is assigned the characteristic shape and position tolerances. A statistical distribution is recorded for each tolerance.
3. Definition of the join conditions: The individual components in an assembly are aligned in sequence. The sequence of the logic links plays a key role in this, because that is how different characteristic chains can be created. Mounting conditions, such as the random utilization of play between a shaft and a bore, can be taken into account.

4. Definition of the measurements: Within the simulation model there is not just one assembly characteristic; virtually any desired number of geometric measurements (distances, gaps, angles, resultant diameters) can be defined as assembly characteristics.

The result of the tolerance analysis is the identified statistical distribution for each assembly characteristic which does not essentially have to be normally distributed. This includes data such as the mean value, standard deviation, expected percentage rejects, and process capability. An additional simulation can also be used to determine the percentage share of each individual tolerance in the distribution of the assembly characteristic (Pareto analysis for contributors).

For more informationen about this see Connect Community 3D Statistical Tolerance Analysis.

5.3. SimTOL®

The SimTOL® program performs tolerance calculations for linear and linearized characteristic chains as well as uncorrelated quantities. It can handle both arithmetic and statistical tolerancing.

The basis for tolerance calculation is a tabular representation of the characteristic chain entered by the user. The table primarily contains the nominal values and the probability densities of the characteristics, their tolerances and sensitivities. Different distribution models can be selected and parameterized.

SimTOL®calculates the tolerance of the assembly characteristic based on numeric convolution operations and outputs its probability density in graphical form. The result can be analyzed in terms of the process capability of the assembly characteristic and the influence of the different contributors, among other criteria.

The program is thus well suited to rapid, user-friendly tolerancing for less complex tolerancing tasks within the development process.

5.4. MATLAB

MATLAB is a commercial software program used to perform mathematical functions and provide graphical representation. MATLAB is deployed especially in development departments to simulate dynamic processes and signal processing algorithms.

With its Statistics Toolbox, MATLAB provides an extensive range of tools for statistical calculation and simulation. Based on those tools, methods for the arithmetic and statistical tolerancing of linear and nonlinear characteristic chains can be implemented, including also statistical simulations and a wide range of evaluation options.

Since programming in MATLAB requires high specialist skills, statistical simulations and evaluations are undertaken in particular when the underlying computation task is being performed in MATLAB anyway.

5.5. Python

The programming language Python provides a standard library whose modules can be used in a program. The module “random” of the standard library allows the generation of pseudo-random numbers, which can be considered as samples of characteristics. The combination of the single values of these samples corresponds finally to a dimension chain. The result of a simulation can be visualized with the function hist (actually matplotlib.pyplot.hist), for example. The computation of mean, standard deviation, variance, or mode is possible with the help of the simple functions contained in the module “statistics”.

6. Annex

6.1. Normal Distribution

Exercise:

Relays were tested for series production start-up. A functionally decisive characteristic is the so-called "response voltage". With 50 relays, the following values of the response voltage U_{an} were measured in volts.

6.2	6.5	6.1	6.3	5.9	6.0	6.0	6.3	6.2	6.4
6.5	5.5	5.7	6.2	5.9	6.5	6.1	6.6	6.1	6.8
6.2	6.4	5.8	5.6	6.2	6.1	5.8	5.9	6.0	6.1
6.0	5.7	6.5	6.2	5.6	6.4	6.1	6.3	6.1	6.6
6.4	6.3	6.7	5.9	6.6	6.3	6.0	6.0	5.8	6.2

a) Calculate $\bar{x}$ and s using a pocket calculator!

b) The customer specifies $U_{an} = 6.8V$. Estimate the proportion of relays which does not satisfy the customer's requirement (fraction nonconforming)!

c) What is the value of response voltage that will be exceeded by maximum 0.1 % of all relays manufactured?

d) Determine with the help of $\bar{x}$, s and the upper specification limit $OGW = 6.8V$ the C_{pk} value of the production!

Solution:

Re a) Mean $\bar{x} = 6.15$, standard deviation $s = 0.2998 \approx 0.3$

Re b) Substitution in the transformation equation results in

$u = \frac{6.8 - 6.15}{0.3} = 2.17$. Table value $\Phi(2.17) = 0.985$, fraction nonconforming $p = 1 - 0.985 = 1.5\%$.

Re c) $p = 1 - \Phi(u) = 0.1\%$ or: $\Phi(u) = 0.999$.

Read from the table that value of u at which $\Phi(u)$ is as close as possible to the value 0.999.

Readout: u = 3.09.

If the equation $u = \frac{x - \bar{x}}{s}$ is resolved for x and the values are substituted, this produces:

$x = \bar{x} + u \cdot s = 6.15 + 3.09 \cdot 0{,}3 \approx 7.1$.

Not more than 0.1 % of all relays will have a response voltage which is in excess of 7.1 V.

Re d) $C_{pk} = \frac{OGW - \bar{x}}{3 \cdot s} = \frac{6.8 - 6.15}{3 \cdot 0.3} \approx 0.7$

6.2. The Probability Plot

If a normal distribution is being spoken of, one usually associates this term with the Gaussian bell curve. The Gaussian bell curve (named for mathematician Carl Friedrich Gauss) is a representation of the probability density *f*(*x*) of the normal distribution:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} . \tag{6.1}$$

The normal distribution states for each value *x* the probability that a random variable *X* will assume a value between -∞ and *x*. One obtains the distribution function *Φ*(*x*) of the normal distribution by integrating the distribution density given above:

$$\Phi(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2} dv . \tag{6.2}$$

Φ(*x*) denotes the area below the Gaussian bell curve up to the value *x*.

The graphical representation of this function follows an s-shaped curve. Strictly speaking, we should always think of this curve, when we talk of a normal distribution.

If the y-axis of this representation is distorted in such a manner, that the s-shaped curve becomes a straight line, a new coordinate system arises: the probability plot. Here the x-axis remains unchanged.

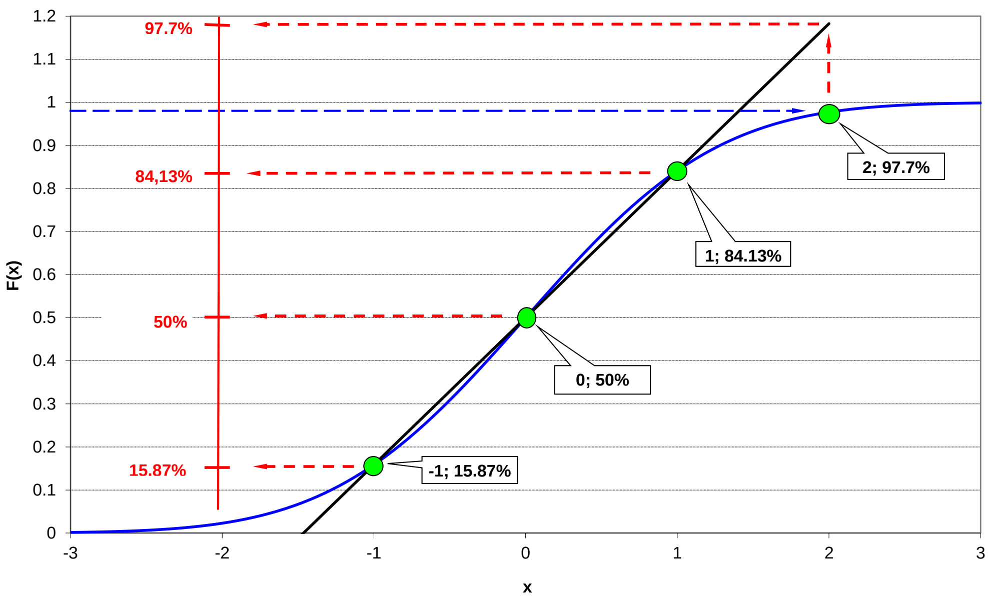


Fig. 6.1: Construction of the probability plot

Because of this correlation, the representation of a normal distribution in this new coordinate system produces a straight line.

This fact is made use of to test a given data set for normality graphically. As long as the number of measurement values is sufficiently big, one can determine the relative frequencies of values within the classes of a grouping and draw a histogram. If the corresponding cumulative relative frequencies are plotted versus the upper class limits on probability paper and a sequence of dots is obtained, which approximately lie on a straight line, then it can be concluded that the values of the data set are approximately normally distributed.

6.3. Standard Normal Distribution

A normally distributed random quantity x with the mean μ and standard deviation σ is converted by the transformation

$$u = \frac{x-\mu}{\sigma} \qquad (6.3)$$

to a likewise normally distributed random quantity u. The mean of U is zero, its standard deviation is one. This special normal distribution is termed standard normal distribution, and is designated $N(0,1)$.

This fact is illustrated by the following figure. Here μ = 23.8 and σ = 0.5. In this example we find:

$$u = \frac{25-23.8}{0.5} = 2.4 \ .$$

The distribution function $\Phi(u)$ indicates the probability with which the random quantity u assumes a value between -∞ and u. $\Phi(u)$ denotes the portion of the area below the Gaussian bell curve up to the value u. The total area under the bell curve has the value one.

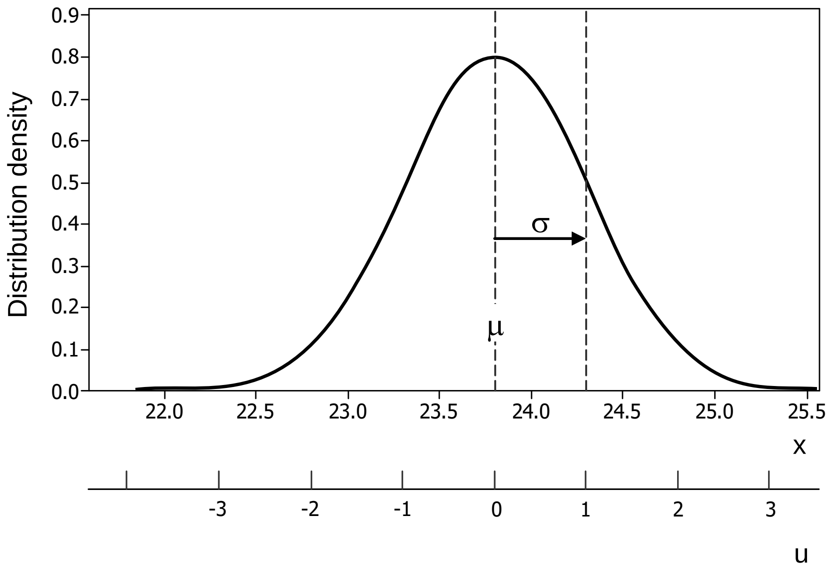


Fig. 6.2: Standard normal distribution

The values for $\Phi(u)$ can be taken from the table. For example, we find $\Phi(2.4)$ = 0.9918. The proportion of all x which are greater than 25 corresponds to the proportion of all u which exceed the value u = 2.4. Accordingly, it is 1 - 0.9918 ≈ 0.8 %.

The following representations show a few examples from which the meaning of the terms and the table values become apparent.

Note that $\Phi(-u) = 1 - \Phi(u)$, $D(u) = \Phi(u) - \Phi(-u)$.

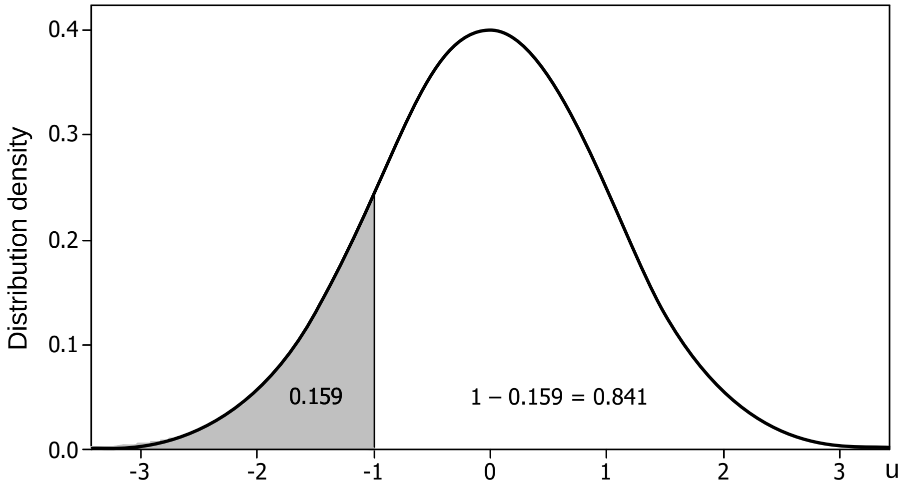


Fig. 6.3: Standard normal distribution with fraction nonconforming 84.1 % (one-sided)

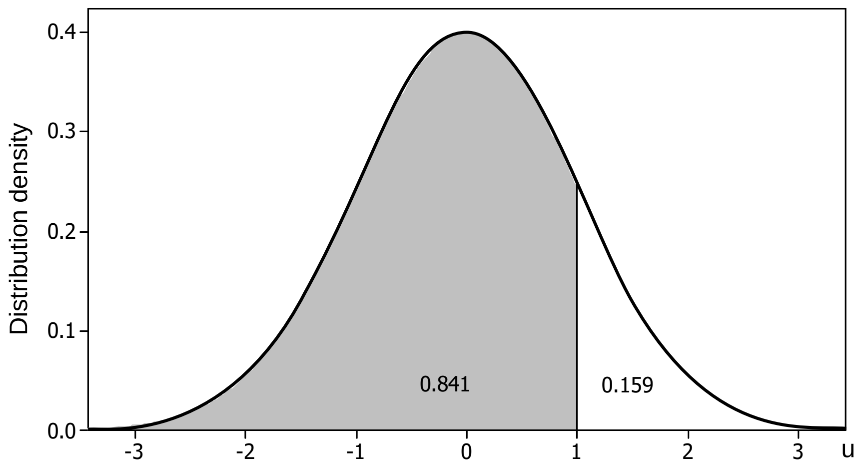


Fig. 6.4: Standard normal distribution with fraction nonconforming 15.9 % (one-sided)

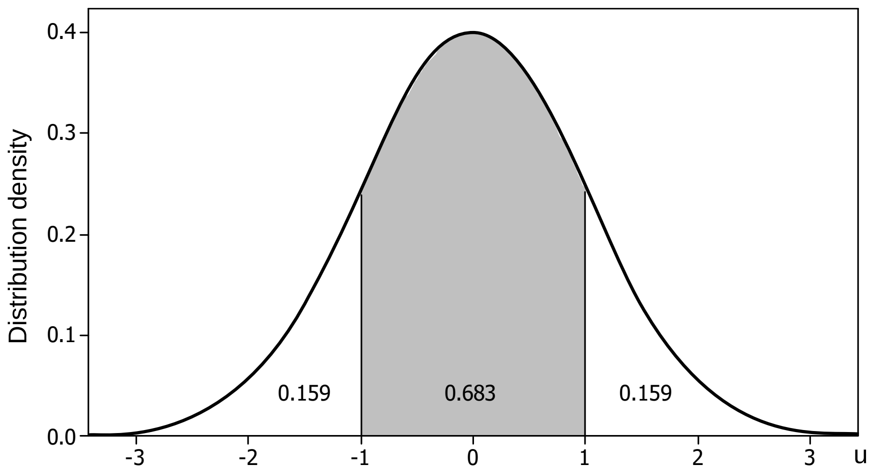


Fig. 6.5: Standard normal distribution with fraction nonconforming 31.8 % (two-sided)

u	Φ(-u)	Φ(u)	D(u)
0,01	0,496011	0,503989	0,007979
0,02	0,492022	0,507978	0,015957
0,03	0,488034	0,511966	0,023933
0,04	0,484047	0,515953	0,031907
0,05	0,480061	0,519939	0,039878
0,06	0,476078	0,523922	0,047844
0,07	0,472097	0,527903	0,055806
0,08	0,468119	0,531881	0,063763
0,09	0,464144	0,535856	0,071713
0,10	0,460172	0,539828	0,079656
0,11	0,456205	0,543795	0,087591
0,12	0,452242	0,547758	0,095517
0,13	0,448283	0,551717	0,103434
0,14	0,444330	0,555670	0,111340
0,15	0,440382	0,559618	0,119235
0,16	0,436441	0,563559	0,127119
0,17	0,432505	0,567495	0,134990
0,18	0,428576	0,571424	0,142847
0,19	0,424655	0,575345	0,150691
0,20	0,420740	0,579260	0,158519
0,21	0,416834	0,583166	0,166332
0,22	0,412936	0,587064	0,174129
0,23	0,409046	0,590954	0,181908
0,24	0,405165	0,594835	0,189670
0,25	0,401294	0,598706	0,197413
0,26	0,397432	0,602568	0,205136
0,27	0,393580	0,606420	0,212840
0,28	0,389739	0,610261	0,220522
0,29	0,385908	0,614092	0,228184
0,30	0,382089	0,617911	0,235823
0,31	0,378280	0,621720	0,243439
0,32	0,374484	0,625516	0,251032
0,33	0,370700	0,629300	0,258600
0,34	0,366928	0,633072	0,266143
0,35	0,363169	0,636831	0,273661
0,36	0,359424	0,640576	0,281153
0,37	0,355691	0,644309	0,288618
0,38	0,351973	0,648027	0,296055
0,39	0,348268	0,651732	0,303463
0,40	0,344578	0,655422	0,310843
0,41	0,340903	0,659097	0,318194
0,42	0,337243	0,662757	0,325515
0,43	0,333598	0,666402	0,332804
0,44	0,329969	0,670031	0,340063
0,45	0,326355	0,673645	0,347290
0,46	0,322758	0,677242	0,354484
0,47	0,319178	0,680822	0,361645
0,48	0,315614	0,684386	0,368773
0,49	0,312067	0,687933	0,375866
0,50	0,308538	0,691462	0,382925

u	Φ(-u)	Φ(u)	D(u)
0,51	0,305026	0,694974	0,389949
0,52	0,301532	0,698468	0,396936
0,53	0,298056	0,701944	0,403888
0,54	0,294599	0,705401	0,410803
0,55	0,291160	0,708840	0,417681
0,56	0,287740	0,712260	0,424521
0,57	0,284339	0,715661	0,431322
0,58	0,280957	0,719043	0,438085
0,59	0,277595	0,722405	0,444809
0,60	0,274253	0,725747	0,451494
0,61	0,270931	0,729069	0,458138
0,62	0,267629	0,732371	0,464742
0,63	0,264347	0,735653	0,471305
0,64	0,261086	0,738914	0,477827
0,65	0,257846	0,742154	0,484308
0,66	0,254627	0,745373	0,490746
0,67	0,251429	0,748571	0,497142
0,68	0,248252	0,751748	0,503496
0,69	0,245097	0,754903	0,509806
0,70	0,241964	0,758036	0,516073
0,71	0,238852	0,761148	0,522296
0,72	0,235762	0,764238	0,528475
0,73	0,232695	0,767305	0,534610
0,74	0,229650	0,770350	0,540700
0,75	0,226627	0,773373	0,546745
0,76	0,223627	0,776373	0,552745
0,77	0,220650	0,779350	0,558700
0,78	0,217695	0,782305	0,564609
0,79	0,214764	0,785236	0,570472
0,80	0,211855	0,788145	0,576289
0,81	0,208970	0,791030	0,582060
0,82	0,206108	0,793892	0,587784
0,83	0,203269	0,796731	0,593461
0,84	0,200454	0,799546	0,599092
0,85	0,197663	0,802337	0,604675
0,86	0,194895	0,805105	0,610211
0,87	0,192150	0,807850	0,615700
0,88	0,189430	0,810570	0,621141
0,89	0,186733	0,813267	0,626534
0,90	0,184060	0,815940	0,631880
0,91	0,181411	0,818589	0,637177
0,92	0,178786	0,821214	0,642427
0,93	0,176186	0,823814	0,647629
0,94	0,173609	0,826391	0,652782
0,95	0,171056	0,828944	0,657888
0,96	0,168528	0,831472	0,662945
0,97	0,166023	0,833977	0,667954
0,98	0,163543	0,836457	0,672914
0,99	0,161087	0,838913	0,677826
1,00	0,158655	0,841345	0,682689

u	Φ(-u)	Φ(u)	D(u)	u	Φ(-u)	Φ(u)	D(u)
1,01	0,156248	0,843752	0,687505	1,51	0,065522	0,934478	0,868957
1,02	0,153864	0,846136	0,692272	1,52	0,064255	0,935745	0,871489
1,03	0,151505	0,848495	0,696990	1,53	0,063008	0,936992	0,873983
1,04	0,149170	0,850830	0,701660	1,54	0,061780	0,938220	0,876440
1,05	0,146859	0,853141	0,706282	1,55	0,060571	0,939429	0,878858
1,06	0,144572	0,855428	0,710855	1,56	0,059380	0,940620	0,881240
1,07	0,142310	0,857690	0,715381	1,57	0,058208	0,941792	0,883585
1,08	0,140071	0,859929	0,719858	1,58	0,057053	0,942947	0,885893
1,09	0,137857	0,862143	0,724287	1,59	0,055917	0,944083	0,888165
1,10	0,135666	0,864334	0,728668	1,60	0,054799	0,945201	0,890401
1,11	0,133500	0,866500	0,733001	1,61	0,053699	0,946301	0,892602
1,12	0,131357	0,868643	0,737286	1,62	0,052616	0,947384	0,894768
1,13	0,129238	0,870762	0,741524	1,63	0,051551	0,948449	0,896899
1,14	0,127143	0,872857	0,745714	1,64	0,050503	0,949497	0,898995
1,15	0,125072	0,874928	0,749856	1,65	0,049471	0,950529	0,901057
1,16	0,123024	0,876976	0,753951	1,66	0,048457	0,951543	0,903086
1,17	0,121000	0,879000	0,757999	1,67	0,047460	0,952540	0,905081
1,18	0,119000	0,881000	0,762000	1,68	0,046479	0,953521	0,907043
1,19	0,117023	0,882977	0,765954	1,69	0,045514	0,954486	0,908972
1,20	0,115070	0,884930	0,769861	1,70	0,044565	0,955435	0,910869
1,21	0,113139	0,886861	0,773721	1,71	0,043633	0,956367	0,912734
1,22	0,111232	0,888768	0,777535	1,72	0,042716	0,957284	0,914568
1,23	0,109349	0,890651	0,781303	1,73	0,041815	0,958185	0,916370
1,24	0,107488	0,892512	0,785025	1,74	0,040930	0,959070	0,918141
1,25	0,105650	0,894350	0,788700	1,75	0,040059	0,959941	0,919882
1,26	0,103835	0,896165	0,792331	1,76	0,039204	0,960796	0,921592
1,27	0,102042	0,897958	0,795915	1,77	0,038364	0,961636	0,923273
1,28	0,100273	0,899727	0,799455	1,78	0,037538	0,962462	0,924924
1,29	0,098525	0,901475	0,802949	1,79	0,036727	0,963273	0,926546
1,30	0,096800	0,903200	0,806399	1,80	0,035930	0,964070	0,928139
1,31	0,095098	0,904902	0,809804	1,81	0,035148	0,964852	0,929704
1,32	0,093418	0,906582	0,813165	1,82	0,034380	0,965620	0,931241
1,33	0,091759	0,908241	0,816482	1,83	0,033625	0,966375	0,932750
1,34	0,090123	0,909877	0,819755	1,84	0,032884	0,967116	0,934232
1,35	0,088508	0,911492	0,822984	1,85	0,032157	0,967843	0,935686
1,36	0,086915	0,913085	0,826170	1,86	0,031443	0,968557	0,937114
1,37	0,085343	0,914657	0,829313	1,87	0,030742	0,969258	0,938516
1,38	0,083793	0,916207	0,832413	1,88	0,030054	0,969946	0,939892
1,39	0,082264	0,917736	0,835471	1,89	0,029379	0,970621	0,941242
1,40	0,080757	0,919243	0,838487	1,90	0,028717	0,971283	0,942567
1,41	0,079270	0,920730	0,841460	1,91	0,028067	0,971933	0,943867
1,42	0,077804	0,922196	0,844392	1,92	0,027429	0,972571	0,945142
1,43	0,076359	0,923641	0,847283	1,93	0,026803	0,973197	0,946393
1,44	0,074934	0,925066	0,850133	1,94	0,026190	0,973810	0,947620
1,45	0,073529	0,926471	0,852941	1,95	0,025588	0,974412	0,948824
1,46	0,072145	0,927855	0,855710	1,96	0,024998	0,975002	0,950004
1,47	0,070781	0,929219	0,858438	1,97	0,024419	0,975581	0,951162
1,48	0,069437	0,930563	0,861127	1,98	0,023852	0,976148	0,952296
1,49	0,068112	0,931888	0,863776	1,99	0,023295	0,976705	0,953409
1,50	0,066807	0,933193	0,866386	2,00	0,022750	0,977250	0,954500

u	Φ(-u)	Φ(u)	D(u)
2,01	0,022216	0,977784	0,955569
2,02	0,021692	0,978308	0,956617
2,03	0,021178	0,978822	0,957643
2,04	0,020675	0,979325	0,958650
2,05	0,020182	0,979818	0,959636
2,06	0,019699	0,980301	0,960601
2,07	0,019226	0,980774	0,961548
2,08	0,018763	0,981237	0,962474
2,09	0,018309	0,981691	0,963382
2,10	0,017864	0,982136	0,964271
2,11	0,017429	0,982571	0,965142
2,12	0,017003	0,982997	0,965994
2,13	0,016586	0,983414	0,966828
2,14	0,016177	0,983823	0,967645
2,15	0,015778	0,984222	0,968445
2,16	0,015386	0,984614	0,969227
2,17	0,015003	0,984997	0,969993
2,18	0,014629	0,985371	0,970743
2,19	0,014262	0,985738	0,971476
2,20	0,013903	0,986097	0,972193
2,21	0,013553	0,986447	0,972895
2,22	0,013209	0,986791	0,973581
2,23	0,012874	0,987126	0,974253
2,24	0,012545	0,987455	0,974909
2,25	0,012224	0,987776	0,975551
2,26	0,011911	0,988089	0,976179
2,27	0,011604	0,988396	0,976792
2,28	0,011304	0,988696	0,977392
2,29	0,011011	0,988989	0,977979
2,30	0,010724	0,989276	0,978552
2,31	0,010444	0,989556	0,979112
2,32	0,010170	0,989830	0,979659
2,33	0,009903	0,990097	0,980194
2,34	0,009642	0,990358	0,980716
2,35	0,009387	0,990613	0,981227
2,36	0,009137	0,990863	0,981725
2,37	0,008894	0,991106	0,982212
2,38	0,008656	0,991344	0,982687
2,39	0,008424	0,991576	0,983152
2,40	0,008198	0,991802	0,983605
2,41	0,007976	0,992024	0,984047
2,42	0,007760	0,992240	0,984479
2,43	0,007549	0,992451	0,984901
2,44	0,007344	0,992656	0,985313
2,45	0,007143	0,992857	0,985714
2,46	0,006947	0,993053	0,986106
2,47	0,006756	0,993244	0,986489
2,48	0,006569	0,993431	0,986862
2,49	0,006387	0,993613	0,987226
2,50	0,006210	0,993790	0,987581

u	Φ(-u)	Φ(u)	D(u)
2,51	0,006037	0,993963	0,987927
2,52	0,005868	0,994132	0,988265
2,53	0,005703	0,994297	0,988594
2,54	0,005543	0,994457	0,988915
2,55	0,005386	0,994614	0,989228
2,56	0,005234	0,994766	0,989533
2,57	0,005085	0,994915	0,989830
2,58	0,004940	0,995060	0,990120
2,59	0,004799	0,995201	0,990402
2,60	0,004661	0,995339	0,990678
2,61	0,004527	0,995473	0,990946
2,62	0,004396	0,995604	0,991207
2,63	0,004269	0,995731	0,991462
2,64	0,004145	0,995855	0,991709
2,65	0,004025	0,995975	0,991951
2,66	0,003907	0,996093	0,992186
2,67	0,003793	0,996207	0,992415
2,68	0,003681	0,996319	0,992638
2,69	0,003573	0,996427	0,992855
2,70	0,003467	0,996533	0,993066
2,71	0,003364	0,996636	0,993272
2,72	0,003264	0,996736	0,993472
2,73	0,003167	0,996833	0,993667
2,74	0,003072	0,996928	0,993856
2,75	0,002980	0,997020	0,994040
2,76	0,002890	0,997110	0,994220
2,77	0,002803	0,997197	0,994394
2,78	0,002718	0,997282	0,994564
2,79	0,002635	0,997365	0,994729
2,80	0,002555	0,997445	0,994890
2,81	0,002477	0,997523	0,995046
2,82	0,002401	0,997599	0,995198
2,83	0,002327	0,997673	0,995345
2,84	0,002256	0,997744	0,995489
2,85	0,002186	0,997814	0,995628
2,86	0,002118	0,997882	0,995764
2,87	0,002052	0,997948	0,995895
2,88	0,001988	0,998012	0,996023
2,89	0,001926	0,998074	0,996148
2,90	0,001866	0,998134	0,996268
2,91	0,001807	0,998193	0,996386
2,92	0,001750	0,998250	0,996500
2,93	0,001695	0,998305	0,996610
2,94	0,001641	0,998359	0,996718
2,95	0,001589	0,998411	0,996822
2,96	0,001538	0,998462	0,996924
2,97	0,001489	0,998511	0,997022
2,98	0,001441	0,998559	0,997118
2,99	0,001395	0,998605	0,997210
3,00	0,001350	0,998650	0,997300

u	Φ(-u)	Φ(u)	D(u)
3,01	0,001306	0,998694	0,997388
3,02	0,001264	0,998736	0,997472
3,03	0,001223	0,998777	0,997554
3,04	0,001183	0,998817	0,997634
3,05	0,001144	0,998856	0,997712
3,06	0,001107	0,998893	0,997787
3,07	0,001070	0,998930	0,997859
3,08	0,001035	0,998965	0,997930
3,09	0,001001	0,998999	0,997998
3,10	0,000968	0,999032	0,998065
3,11	0,000935	0,999065	0,998129
3,12	0,000904	0,999096	0,998191
3,13	0,000874	0,999126	0,998252
3,14	0,000845	0,999155	0,998311
3,15	0,000816	0,999184	0,998367
3,16	0,000789	0,999211	0,998422
3,17	0,000762	0,999238	0,998476
3,18	0,000736	0,999264	0,998527
3,19	0,000711	0,999289	0,998577
3,20	0,000687	0,999313	0,998626
3,21	0,000664	0,999336	0,998673
3,22	0,000641	0,999359	0,998718
3,23	0,000619	0,999381	0,998762
3,24	0,000598	0,999402	0,998805
3,25	0,000577	0,999423	0,998846
3,26	0,000557	0,999443	0,998886
3,27	0,000538	0,999462	0,998925
3,28	0,000519	0,999481	0,998962
3,29	0,000501	0,999499	0,998998
3,30	0,000483	0,999517	0,999033
3,31	0,000466	0,999534	0,999067
3,32	0,000450	0,999550	0,999100
3,33	0,000434	0,999566	0,999132
3,34	0,000419	0,999581	0,999162
3,35	0,000404	0,999596	0,999192
3,36	0,000390	0,999610	0,999221
3,37	0,000376	0,999624	0,999248
3,38	0,000362	0,999638	0,999275
3,39	0,000349	0,999651	0,999301
3,40	0,000337	0,999663	0,999326
3,41	0,000325	0,999675	0,999350
3,42	0,000313	0,999687	0,999374
3,43	0,000302	0,999698	0,999396
3,44	0,000291	0,999709	0,999418
3,45	0,000280	0,999720	0,999439
3,46	0,000270	0,999730	0,999460
3,47	0,000260	0,999740	0,999480
3,48	0,000251	0,999749	0,999499
3,49	0,000242	0,999758	0,999517
3,50	0,000233	0,999767	0,999535

u	Φ(-u)	Φ(u)	D(u)
3,51	0,000224	0,999776	0,999552
3,52	0,000216	0,999784	0,999568
3,53	0,000208	0,999792	0,999584
3,54	0,000200	0,999800	0,999600
3,55	0,000193	0,999807	0,999615
3,56	0,000185	0,999815	0,999629
3,57	0,000178	0,999822	0,999643
3,58	0,000172	0,999828	0,999656
3,59	0,000165	0,999835	0,999669
3,60	0,000159	0,999841	0,999682
3,61	0,000153	0,999847	0,999694
3,62	0,000147	0,999853	0,999705
3,63	0,000142	0,999858	0,999717
3,64	0,000136	0,999864	0,999727
3,65	0,000131	0,999869	0,999738
3,66	0,000126	0,999874	0,999748
3,67	0,000121	0,999879	0,999757
3,68	0,000117	0,999883	0,999767
3,69	0,000112	0,999888	0,999776
3,70	0,000108	0,999892	0,999784
3,71	0,000104	0,999896	0,999793
3,72	0,000100	0,999900	0,999801
3,73	0,000096	0,999904	0,999809
3,74	0,000092	0,999908	0,999816
3,75	0,000088	0,999912	0,999823
3,76	0,000085	0,999915	0,999830
3,77	0,000082	0,999918	0,999837
3,78	0,000078	0,999922	0,999843
3,79	0,000075	0,999925	0,999849
3,80	0,000072	0,999928	0,999855
3,81	0,000069	0,999931	0,999861
3,82	0,000067	0,999933	0,999867
3,83	0,000064	0,999936	0,999872
3,84	0,000062	0,999938	0,999877
3,85	0,000059	0,999941	0,999882
3,86	0,000057	0,999943	0,999887
3,87	0,000054	0,999946	0,999891
3,88	0,000052	0,999948	0,999896
3,89	0,000050	0,999950	0,999900
3,90	0,000048	0,999952	0,999904
3,91	0,000046	0,999954	0,999908
3,92	0,000044	0,999956	0,999911
3,93	0,000042	0,999958	0,999915
3,94	0,000041	0,999959	0,999919
3,95	0,000039	0,999961	0,999922
3,96	0,000037	0,999963	0,999925
3,97	0,000036	0,999964	0,999928
3,98	0,000034	0,999966	0,999931
3,99	0,000033	0,999967	0,999934
4,00	0,000032	0,999968	0,999937

6.4. Handling of Correlated Quantities

In the previous sections the assembly characteristic tolerance was defined for the case that the individual characteristics are statistically independent of each other. This precondition is not generally met, however, so it becomes necessary to extend the method for correlated characteristics [5].

As the statistical measure of the interdependence of characteristics, the correlation coefficient r of a random sample is:

$$r = \frac{s_{xy}}{s_x s_y} = \frac{\sum_{i=1}^{n}\left((x_i - \bar{x})(y_i - \bar{y})\right)}{\sqrt{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}\sqrt{\sum_{i=1}^{n}\left(y_i - \bar{y}\right)^2}}, \tag{6.4}$$

and the correlation coefficient ρ of the underlying population

$$\rho = \frac{\sigma_{xy}}{\sigma_x \sigma_y}. \tag{6.5}$$

The tolerance overlay of correlated tolerance influences is based on these characteristic quantities. The correlation coefficient ρ of the population is often not known. So it is estimated by way of the correlation coefficient r of the sample.

$$\frac{\sigma_{xy}}{\sigma_x \sigma_y} = \rho \approx r = \frac{s_{xy}}{s_x s_y}. \tag{6.6}$$

6.4.1. Correlation in Case of Linear Characteristic Chains

The variance of a sum of correlated random numbers can be calculated by:

$$\sigma_z^2 = \sigma_{z1}^2 + \sigma_{z2}^2 + 2\sigma_{z1z2}. \tag{6.7}$$

As the individual tolerances z_i are produced from individual characteristics x_i and the respective sensitivities E_{xi}, the tolerance of the assembly characteristic can be calculated with

$$\sigma_{z1z2} = E_{x1} E_{x2} \sigma_{x1x2} \tag{6.8}$$

as:

$$\sigma_z^2 = \sigma_{z1}^2 + \sigma_{z2}^2 + 2\sigma_{z1z2} = E_{x1}^2 \sigma_{x1}^2 + E_{x2}^2 \sigma_{x2}^2 + 2E_{x1} E_{x2} \sigma_{x1x2}. \tag{6.9}$$

If two individual characteristics x_1 and x_2 have a correlation ρ, because of

$$\sigma_{x1x2} = \rho \sigma_{x1} \sigma_{x2} \tag{6.10}$$

the variance of the assembly characteristic is produced as:

$$\sigma_z^2 = E_{x1}^2 \sigma_{x1}^2 + E_{x2}^2 \sigma_{x2}^2 + 2E_{x1} E_{x2} \rho \sigma_{x1} \sigma_{x2}. \tag{6.11}$$

The correlation of the individual characteristic x_1 and x_2 may have a positive or negative effect on the assembly tolerance.

6.4.2. Correlation in Case of Nonlinear Characteristic Chains

In the case of nonlinear characteristic chains, a linearization at the operating point may be affected so that the tolerance calculation is traced back to linear characteristic chains. If linearization is not possible due to a high degree of nonlinearity, a statistical simulation can be carried out. However, in contrast to the tolerance simulation described, when generating the random numbers the correlation of individual characteristics has to be taken into account.

To generate two correlated characteristics x_1 and x_2, two uncorrelated quantities w_1 and w_2 are first generated with a standard deviation $\sigma_1 = \sigma_2 = 1$ and $\mu_1 = \mu_2 = 0$. By means of a transformation:

$$x_1 = \sigma_{x1} w_1 \,,\; x_2 = \rho \sigma_{x2} w_1 + \sqrt{1-\rho^2}\,\sigma_{x2} w_2 \tag{6.12}$$

two characteristics x_1 and x_2 are produced with the standard deviations σ_{x1} and σ_{x2} as well as the mean values $\mu_{x1} = 0$ and $\mu_{x2} = 0$. They have a covariance of:

$$\sigma_{x1x2} = E\big((x_1 - \mu_{x1})(x_2 - \mu_{x2})\big) = E(x_1 x_2) = \rho \sigma_{x1} \sigma_{x2} \tag{6.13}$$

and thus, have the correlation ρ. The mean values μ_{x1} and μ_{x2} are added to the random variables as appropriate. With the random variables generated in this way, a statistical tolerance simulation as described in section 0 is carried out.

6.4.3. Example: Statistical Tolerancing of Correlated Individual Characteristics

For the example of the voltage divider from Figure 4.8 it is assumed that the two resistors are integrated together in one circuit. They are therefore manufactured in the same production process, and it can be assumed that the two resistors have a correlation ρ close to 1. The tolerance influence of the voltage source is further assumed as statistically independent. Thus, the tolerance of the output voltage U following linearization results as:

$$\sigma_U^2 = E_{R1}^2 \sigma_{R1}^2 + E_{R2}^2 \sigma_{R2}^2 + 2\rho E_{R1} \sigma_{R1} E_{R2} \sigma_{R2} + E_{Uref}^2 \sigma_{Uref}^2 \,. \tag{6.14}$$

For the resistances a normal distribution with mean value $R_0 = 100\ \Omega$ and a standard deviation of $\sigma_R = 1\ \Omega$ is assumed. Then the sensitivities E_{R1} and E_{R2} are equal in amount, but are signed differently

$$E_R = -E_{R1} = E_{R2} \,. \tag{6.15}$$

The above equation can thus be simplified to:

$$\begin{aligned} \sigma_U^2 &= E_{R1}^2 \sigma_{R1}^2 + E_{R2}^2 \sigma_{R2}^2 + 2\rho E_{R1} \sigma_{R1} E_{R2} \sigma_{R2} + E_{Uref}^2 \sigma_{Uref}^2 \\ &= E_R^2 \sigma_R^2 + E_R^2 \sigma_R^2 - 2\rho E_R \sigma_R E_R \sigma_R + E_{Uref}^2 \sigma_{Uref}^2 \\ &= 2E_R^2 \sigma_R^2 (1-\rho) + E_{Uref}^2 \sigma_{Uref}^2 \end{aligned} \tag{6.16}$$

It becomes clear that as the correlation coefficient ρ increases the variance of the assembly characteristic decreases. In the ideal case, both resistances are identical, resulting in a correlation coefficient $\rho = 1$. In this case the absolute value of the resistance has no influence on the variance of the assembly characteristic.

The effect of the tolerance compensation can also be read from the transfer function:

$$U = \frac{R_2}{R_1 + R_2} U_{ref} = \frac{1.1 R_2}{1.1 R_1 + 1.1 R_2} U_{ref} \tag{6.17}$$

If all components have a 10 % higher value, the factor 1.1 is shortened out of the fraction. The absolute value of the resistances is not relevant to the function; only the resistance ratio is of importance for the output voltage.

6.5. Simulation of Statistical Distributions in EXCEL

Due to the central limit theorem of statistics the assembly dimension will be approximately normally distributed, independently of the distribution of the individual dimensions.

Such considerations often trigger the wish to mathematically simulate the assembly of individual components. This can easily be done with EXCEL.

Each dimension in the stack is assigned to a column.

Using the formulas contained in the following sections, one generates a sequence of random numbers in each column according to the distribution chosen for that feature

"Linking" the rows within a column yields values for the gap. These links can be simple addition (taking the appropriate counting direction into account), but more complicated functional relationships of the individual dimensions can also be simulated.

This type of simulation is not limited to statistical tolerancing. Application to production processes and measurement systems, or to reliability technology (Weibull theory) is also conceivable.

6.5.1. Pseudo-random numbers

With the help of the function "RAND()" in EXCEL a sequence of values equally distributed in the interval [0, 1] can be generated. For this purpose, "Automatic" must be set in the "Formulas" tab under "Calculation options". After pressing the F9 key or deleting an empty cell, a new calculation is then performed. The seemingly random sequence are so-called pseudo-random numbers, which are generated according to a deterministic process, e.g. the multiplicative congruence method or the newer Mersenne-Twister algorithm. It is inevitable that the generated sequence of numbers repeats after a certain number of numbers. However, the period length is very large.

6.5.2. Selection of the distribution

These simulations are often performed during development work, in order to determine whether or not, or how often, the design could cause problems in later machining or assembly (e.g. difficulties in joining two components).

The biggest problem in mathematical modelling of the tolerance stack is usually to estimate how the individual dimensions will be distributed in later production. If you look at SPC control charts from real processes, or histograms of this data, you will recognize that the normal distribution is often only a rough approximation.

In particular, dimensions limited by zero (e.g. form or position dimensions) can only have values larger than zero and thus automatically have a right-skewed distribution with zero as the lower limit.

Usually, linking normally distributed dimensions results in a "good" distribution of the gap, i.e. the resulting distribution is relatively narrow compared to the gap tolerance.

On the other hand, if a rectangular distribution is used for the individual dimensions, then the resulting distribution is much wider. The rectangular distribution is thus a more pessimistic assumption.

The user usually has no other options than making the most realistic assumptions possible on the expected production distributions, together with the manufacturing engineer.

Of course, appropriate experience with similar processes already in production can be helpful. However, it is not sufficient to simply consider the Cp or Cpk values since these indices do not indicate the type of distribution of the dimension.

6.5.3. Rectangular distribution

A uniformly distributed (rectangularly distributed) random number in an interval [0; 1] can be generated in EXCEL with the command “= RAND()”. If the rectangular distribution in the interval [LSL, USL] is desired, then simply multiply the random number by the interval width and add the LSL:

= LSL + (USL-LSL)*RAND()

If the LSL is in cell C5 and the USL in cell C7, then the corresponding EXCEL formula is:

=C$5+(C$7-C$5)* RAND()

6.5.4. Normal distribution

Normally distributed random numbers x can be generated with the following command:

=NORMINV(RAND();B$5;B$7)

In this example, the average μ s in cell B5 and the standard deviation σ in cell B7.

6.5.5. Logarithmic normal distribution

Measured values can be described with a logarithmic normal distribution if the logarithms of these values are normally distributed. In the reverse sense, we can obtain logarithmic normal-ly distributed numbers by applying this function to normally distributed numbers (as obtained from the formula in Section 2):

=EXP(NORMINV(RAND(); B$5; B$7))

In this example, the average μ is in cell B5 and the standard deviation σ in cell B7.

6.5.6. Triangular distribution

A triangular distribution is obtained by adding two rectangular distributions. It is obtained by generating two random numbers with the formula in Section 1 and calculating their average.

= LSL + (USL-LSL)*(RAND()+RAND())*0.5

With LSL in cell D5 and USL in D7, the EXCEL formula is:

=D$5+(D$7-D$5)*(RAND()+RAND())*0.5

6.5.7. Trapezoidal distribution

A trapezoidal distribution is obtained by adding two rectangular distributions of different widths.

6.5.8. U-shaped distribution

A U-shaped distribution can result from periodic events.

This distribution can be simulated over an interval [LSL, USL] using the following function:

$$x = \frac{USL+LSL}{2} + \frac{USL-LSL}{2} \cdot sin\left(\frac{\pi}{2} \cdot (-1 + 2 \cdot RAND())\right)$$

The term $-1 + 2 \cdot RAND()$ generates a uniformly distributed random number in the interval [-1; +1], or if multiplied by $\frac{\pi}{2}$, a uniformly distributed random number in the interval $\left[-\frac{\pi}{2};\ +\frac{\pi}{2}\right]$.

Multiplying by the interval width $\frac{USL-LSL}{2}$ and addition of the middle of the interval $\frac{USL+LSL}{2}$ (which is equivalent to a shift) delivers the desired distribution.

With LSL in cell D5 and USL in D7, the EXCEL formula is:

=0,5*(E$7+E$5)+0,5*(E$7-E$5)*SIN(0,5*PI()*(-1+2* RAND()))

6.5.9. Weibull distribution

If we interpret the uniformly distributed random numbers generated with a random number generator in an interval [0,1] as the values of the cumulative relative frequency $F(t)$ and insert them in the following equation

$t = T \cdot \left[ln\left(\frac{1}{1-F(t)}\right)\right]^{\frac{1}{b}}$ then we obtain the Weibull distributed failure times t,

with b and T as parameters of the Weibull distribution.

If b is in cell A4 and T in cell A2, the EXCEL formula is:

=A$2*POWER(LN(1/(1-RAND()));1/A$4)

6.5.10. Binomial distribution

If the number of trials n (the sample size) is entered in cell B3 and the probability of success p in B4, then binomially distributed random numbers are generated with

=KRITBINOM(B$3; B$4; RAND())

6.5.11. Two-dimensional normal distribution

The position (e.g. of a bore) in a plane can be indicated with the x and y coordinates with respect to a reference point (the “origin” of the coordinate system). If x and y are considered as independent random variables and if they are both simulated with a normal distribution (refer to Section 2), then we obtain a two dimensional normally distributed random variable for the position. On the other hand, if we use a rectangular distribution for x and y, then we obtain a random variable that varies within a square or rectangle in a plane.

However, position tolerances in a plane are normally specified within a circle with a certain diameter d. Thus, it is better to use polar coordinates for the simulation. If the angle is rectangularly distributed in the interval $[0; 2\pi]$ and the radius is also rectangularly distributed in the interval [0; d/2], then we obtain the desired random variable for the position, which is uniformly distributed within a circle of diameter d.

6.5.12. Random numbers subject to a given empirical distribution

The following solution describes a possibility to realize this in the framework of a VBA subprogram (macro) which is programmed in a VBA module. This procedure uses the single known relative frequencies h(i) of the empirical distribution on hand (histogram).

- Step 1: VBA command rnd(1) creates a uniformly distributed random number from the interval [0; 1]. Multiplication with upper limit ("maxvalue") of the outmost right class results in a random number "avalue" from the desired range.
- Step 2: It is determined to which class of the empirical distribution "avalue" belongs.
- Step 3: With rnd(1) a second random number "bvalue" from [0; 1] is generated.
- Step 4: The decision is made whether "avalue" is accepted or not. This is the case if bvalue < h(class). Thus, the random numbers allocated to this class can only appear with frequency h(class).
- In a loop, steps 1 to 4 are repeated until "avalue" is accepted in step 4.
- Step 5: Finally, the command "cells(line, column)" writes random number x = avalue in the corresponding cell of the table.

The whole procedure is repeated according to the desired number of random numbers.

7. Literature

[1] B. Klein: Prozessorientierte Statistische Tolerierung im Maschinen- u. Fahrzeugbau, Expert-Verlag, 2014 (German)

[2] G. Kirschling: Qualitätssicherung und Toleranzen, Springer-Verlag Berlin, 1988 (German)

[3] Series Quality Management in the Bosch Group, Booklet No. 7: Statistical Process Control SPC

[4] DIN EN ISO 286-1:2019-09: Geometrical product specification (GPS) – ISO tolerance system for lengths – Part 1

DIN ISO 3534-1:2009-10: Statistics - Vocabulary and symbols - Part 1: General statistical terms and terms used in probability

[5] M. Strohrmann: Design For Six Sigma Skriptum, www.home.hs-karlsruhe.de/~stma0003, Access Feb. 1, 2022

[6] Series Quality Management in the Bosch Group, Booklet No. 9: Machine and Process Capability

8. Index

Robert Bosch GmbH

C/QMM Tilsch

Wiener Straße 42 - 46

70469 Stuttgart

Germany

Telefon +49 711 811 - 0
www.bosch.com

Robert Bosch GmbH
C/QMM Tilsch
Wiener Strasse 42 - 46
70469 Stuttgart
Germany
Phone +49 711 811 - 0
www.bosch.com